On the trivectors of a 6-dimensional symplectic vector space. IV

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Abstract

Let V be a 6-dimensional vector space over a field $\mathbb{F}$ equipped with a nondegenerate alternating bilinear form f. The group $GL(V)$ has a natural action on the third exterior power $\bigwedge^3 V$ of V which defines five families of nonzero trivectors of V. Four of these families are always orbits regardless of the structure of the underlying field $\mathbb{F}$. The orbits contained in the fifth family are in one-to-one correspondence with the quadratic extensions of $\mathbb{F}$ that are contained in a fixed algebraic closure of $\mathbb{F}$. We will divide those orbits corresponding to the nonseparable quadratic extensions into suborbits for the action of the symplectic group $Sp(V,f) \cong Sp_6(\mathbb{F})$ associated with (V,f) on $\bigwedge^3 V$.

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1 Introduction

Throughout this paper, let V denote a 6-dimensional vector space over a field $\mathbb{F}$ and let f denote a nondegenerate alternating bilinear form on V. With the pair (V,f), there is associated a symplectic group $Sp(V,f) \cong Sp_6(\mathbb{F})$ which consists of all $\theta \in GL(V)$ for which $f(\bar{x}^\theta, \bar{y}^\theta) = f(\bar{x}, \bar{y})$, $\forall \bar{x}, \bar{y} \in V$. The group $Sp(V,f)$ consists of those elements of $GL(V)$ that map hyperbolic bases of (V,f) to hyperbolic bases of (V,f). With a hyperbolic basis of (V,f) we mean an ordered basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of V such that $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = \delta_{ij}$ for all $i, j \in \{1,2,3\}$. Here, δ_{ij} denotes the Kronecker delta.

The group $GL(V)$ and its subgroup $Sp(V,f)$ have a natural action on the third exterior power $\bigwedge^3 V$ of V. Indeed, for every $\theta \in GL(V)$, there exists a unique $\bigwedge^3(\theta) \in GL(\bigwedge^3 V)$ such that $\bigwedge^3(\theta)(\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3) = \theta(\bar{v}_1) \wedge \theta(\bar{v}_2) \wedge \theta(\bar{v}_3)$, $\forall \bar{v}_1, \bar{v}_2, \bar{v}_3 \in V$. The elements of $\bigwedge^3 V$ are called the *trivectors* of V. Two trivectors χ_1 and χ_2 of V are called *G-equivalent*, where G is either $GL(V)$ or $Sp(V,f)$, if there exists a $\theta \in G$ such that $\bigwedge^3(\theta)(\chi_1) = \chi_2$.

Let $\overline{\mathbb{F}}$ be a fixed algebraic closure of $\mathbb{F}$. For every quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$, there exist $\mu_{\mathbb{F}'}, \lambda_{\mathbb{F}'} \in \mathbb{F}^* := \mathbb{F} \setminus \{0\}$ such that $p_{\mathbb{F}'}(X) := \mu_{\mathbb{F}'}X^2 - (\mu_{\mathbb{F}'}\lambda_{\mathbb{F}'} + \mu_{\mathbb{F}'} + \lambda_{\mathbb{F}'})X + \lambda_{\mathbb{F}'}$ is an irreducible quadratic polynomial of $\mathbb{F}[X]$ and $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ defined by $p_{\mathbb{F}'}(X)$. We define $\chi^*_{\mathbb{F}'} := \mu_{\mathbb{F}'} \cdot \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \lambda_{\mathbb{F}'} \cdot \bar{v}_4 \wedge \bar{v}_5 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_4) \wedge (\bar{v}_2 + \bar{v}_5) \wedge (\bar{v}_3 + \bar{v}_6)$, where $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ is some basis of V. Different choices for $\mu_{\mathbb{F}'}$, $\lambda_{\mathbb{F}'}$ and $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6)$ usually give rise to distinct trivectors, but as shown in De Bruyn [3], the $GL(V)$-equivalence class of the trivector $\chi^*_{\mathbb{F}'}$ only depends on $\mathbb{F}'$ and not on the particular choices of $\mu_{\mathbb{F}'}$, $\lambda_{\mathbb{F}'}$ and $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6)$.

A complete classification of all $GL(V)$-equivalence classes of trivectors of V was obtained in Revoy [10].

Proposition 1.1 ([10]) *Let $\{\bar{v}^*_1, \bar{v}^*_2, \ldots, \bar{v}^*_6\}$ be a fixed basis of V. Then every nonzero trivector of V is $GL(V)$-equivalent with precisely one of the following trivectors:*

(A) $\bar{v}^*_1 \wedge \bar{v}^*_2 \wedge \bar{v}^*_3$*;*

(B) $\bar{v}^*_1 \wedge \bar{v}^*_2 \wedge \bar{v}^*_3 + \bar{v}^*_1 \wedge \bar{v}^*_4 \wedge \bar{v}^*_5$*;*

(C) $\bar{v}^*_1 \wedge \bar{v}^*_2 \wedge \bar{v}^*_3 + \bar{v}^*_4 \wedge \bar{v}^*_5 \wedge \bar{v}^*_6$*;*

(D) $\bar{v}^*_1 \wedge \bar{v}^*_2 \wedge \bar{v}^*_4 + \bar{v}^*_2 \wedge \bar{v}^*_3 \wedge \bar{v}^*_5 + \bar{v}^*_3 \wedge \bar{v}^*_1 \wedge \bar{v}^*_6$*;*

(E) $\chi^*_{\mathbb{F}'}$ *for some quadratic extension* $\mathbb{F}'$ *of* $\mathbb{F}$ *contained in* $\overline{\mathbb{F}}$*.*

Other classification results for $GL(V)$-equivalence classes of trivectors of V, valid for certain classes of fields, can also be found in Cohen & Helminck [1] and Reichel [9]. A nonzero trivector of V is said to be of *Type* $(X) \in \{(A), (B), (C), (D), (E)\}$ if it is $GL(V)$-equivalent with (one of) the trivector(s) described in (X) of Proposition 1.1. The description of the trivectors of Type (E) given above is different from the ones given in [1] and [10]. The above description is taken from [3].

With the pair (V, f), there is associated a symplectic dual polar space $DW(5, \mathbb{F})$. This is the point-line geometry whose points [resp., lines] are the totally isotropic 3-spaces [resp., totally isotropic 2-spaces] of V (with respect to f), with incidence being reverse containment. A *hyperplane* H of a point-line geometry $\mathcal{S}$ is a set of points, distinct from the whole point set, having the property that every line of $\mathcal{S}$ has either precisely one or all its points in H. The knowledge of the $Sp(V, f)$-equivalence classes of trivectors of V is important for the classification and study of the hyperplanes of $DW(5, \mathbb{F})$ arising from its so-called Grassmann embedding, as well as the study of the relationships between the hyperplanes of $DW(5, \mathbb{F})$ and the hyperplanes of the plane Grassmannian of PG(V). The latter is a certain point-line geometry into which $DW(5, \mathbb{F})$ is fully embeddable.

Popov [8, Section 3] obtained a complete classification of all $Sp(V, f)$-equivalence classes of trivectors of V, assuming the underlying field $\mathbb{F}$ is algebraically closed and of characteristic distinct from 2. Popov's method heavily relies on the decomposition of $\bigwedge^3 V$ as a direct sum of two subspaces Ω_1 and Ω_2 of respective dimensions 14 and 6 (which are also submodules for the action of $Sp(V, f)$ on $\bigwedge^3 V$). This decomposition is only valid for fields of characteristic distinct from 2. Popov's proof also relies on a result of Igusa [7] regarding the $Sp(V, f)$-equivalence classes of trivectors contained in Ω_1. This classification result of

Igusa also assumes that the underlying field is algebraically closed and of characteristic distinct from 2. We are interested in the classification of all $Sp(V,f)$-equivalence classes of trivectors without imposing any restrictions on the underlying field $\mathbb{F}$. Such classification results were already obtained in De Bruyn & Kwiatkowski [4] for trivectors of Type (A), (B), (C), in De Bruyn & Kwiatkowski [6] for trivectors of Type (D) and in De Bruyn & Kwiatkowski [5] for trivectors of Type (E) under the additional assumption that the corresponding quadratic field extension is separable. The present paper deals with those trivectors of Type (E) whose corresponding quadratic field extensions are nonseparable. More precisely, we will solve the following problem.

> Let $\mathbb{F}'$ be a fixed nonseparable quadratic extension of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$. Let $\mathcal{E}_{\mathbb{F}'}$ denote the set of all trivectors of V which are $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$. Then determine the $Sp(V,f)$-equivalence classes into which $\mathcal{E}_{\mathbb{F}'}$ splits.

Since $\mathbb{F}'$ is a nonseparable quadratic extension of $\mathbb{F}$, the characteristic of $\mathbb{F}$ is 2 and there exists a nonsquare a in $\mathbb{F}$ such that $\mathbb{F}'$ is the extension of $\mathbb{F}$ defined by the irreducible quadratic polynomial X^2+a. Let $(\bar{e}_1^*, \bar{f}_1^*, \bar{e}_2^*, \bar{f}_2^*, \bar{e}_3^*, \bar{f}_3^*)$ be a fixed hyperbolic basis of (V,f).

For all $h_1,h_2,h_3 \in \mathbb{F}^*$, we define

$$\begin{aligned}\chi_1(h_1,h_2,h_3) \ &:= \ \frac{a+1}{a}\cdot \bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{e}_3^* + (a+1)h_1h_2h_3\cdot \bar{f}_1^*\wedge\bar{f}_2^*\wedge\bar{f}_3^* \\ &\qquad + (\bar{e}_1^*+h_1\bar{f}_1^*)\wedge(\bar{e}_2^*+h_2\bar{f}_2^*)\wedge(\bar{e}_3^*+h_3\bar{f}_3^*).\end{aligned}$$

Any trivector of V which is $Sp(V,f)$-equivalent with a trivector of the form $\chi_1(h_1,h_2,h_3)$ for some $h_1,h_2,h_3\in\mathbb{F}^*$ is called a *trivector of Type (E1')*.

For every $k\in\mathbb{F}^*$ and all $h_1,h_2\in\mathbb{F}$ satisfying $h_1h_2(a+1)^2\neq 1$, we define the following trivector of V:

$$\begin{aligned}\chi_2(k,h_1,h_2) \ &:= \ \frac{1}{a}\cdot \bar{e}_1^*\wedge(\bar{e}_2^*+h_1(a+1)\bar{f}_3^*)\wedge\bar{f}_2^* + k\cdot\bar{f}_1^*\wedge\bar{e}_3^*\wedge(h_2(a+1)\bar{e}_2^*+\bar{f}_3^*) \\ &\qquad + \frac{1}{(a+1)^2}\cdot(\bar{e}_1^*+k\bar{f}_1^*)\wedge(\bar{e}_2^*+(a+1)\bar{e}_3^*+h_1(a+1)\bar{f}_3^*)\wedge(h_2(a+1)\bar{e}_2^* \\ &\qquad + (a+1)\bar{f}_2^*+\bar{f}_3^*).\end{aligned}$$

Any trivector of V which is $Sp(V,f)$-equivalent with a trivector of the form $\chi_2(k,h_1,h_2)$ for some $k,h_1,h_2\in\mathbb{F}$ satisfying $k\neq 0$ and $h_1h_2(a+1)^2\neq 1$ is called a *trivector of Type (E2')*.

For all $h_1,h_2\in\mathbb{F}$ with $h_1\neq 0$, we define

$$\begin{aligned}\chi_3(h_1,h_2) \ &:= \ \frac{1}{a}\cdot\bar{e}_1^*\wedge(\bar{e}_2^*+\bar{e}_3^*)\wedge\bar{f}_2^* + \bar{e}_2^*\wedge\bar{f}_1^*\wedge(\bar{e}_1^*+h_1\bar{f}_3^*) \\ &\qquad + \frac{1}{a+1}\cdot(\bar{e}_1^*+\bar{e}_2^*)\wedge(\bar{e}_3^*+h_1\bar{f}_3^*)\wedge\Big((a+1)^2h_2\bar{e}_1^*+\bar{f}_1^*+\bar{f}_2^*\Big).\end{aligned}$$

Any trivector of V which is $Sp(V,f)$-equivalent with a trivector of the form $\chi_3(h_1,h_2)$ for some $h_1\in\mathbb{F}^*$ and some $h_2\in\mathbb{F}$ is called a *trivector of Type (E3')*.

The following two theorems are the main results of this paper.

Theorem 1.2 *The trivectors of V that are $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ are precisely the trivectors of Type $(E1')$, $(E2')$ and $(E3')$.*

Theorem 1.3 (1) *Let $i, j \in \{1, 2, 3\}$ with $i \neq j$. Then no trivector of Type (Ei') is $Sp(V, f)$-equivalent with a trivector of Type (Ej').*

(2) *Let $h_1, h_2, h_3, h'_1, h'_2, h'_3 \in \mathbb{F}^*$. Then the two trivectors $\chi_1(h_1, h_2, h_3)$ and $\chi_1(h'_1, h'_2, h'_3)$ of V are $Sp(V, f)$-equivalent if and only if there exists a matrix $A \in SL(3, \mathbb{F}')$ such that* $\mathrm{diag}(h'_1, h'_2, h'_3) = A \cdot \mathrm{diag}(h_1, h_2, h_3) \cdot A^{\mathrm{T}}$.

(3) *Let $k, h_1, h_2, k', h'_1, h'_2 \in \mathbb{F}$ with $k \neq 0 \neq k'$ and $h_1h_2(a+1)^2 \neq 1 \neq h'_1h'_2(a+1)^2$. Then the two trivectors $\chi_2(k, h_1, h_2)$ and $\chi_2(k', h'_1, h'_2)$ of V are $Sp(V, f)$-equivalent if and only if $k = k'$, $h_1h_2 = h'_1h'_2$ and there exist $X, Y, Z, U \in \mathbb{F}$ such that $h'_1 = h_1(X^2 + aY^2) + h_2(Z^2 + aU^2) + (XU + YZ)$.*

(4) *Let $h_1, h_2, h'_1, h'_2 \in \mathbb{F}$ with $h_1 \neq 0 \neq h'_1$. Then the two trivectors $\chi_3(h_1, h_2)$ and $\chi_3(h'_1, h'_2)$ of V are $Sp(V, f)$-equivalent if and only if $h_1 = h'_1$ and $h_2 + h'_2$ is of the form $h_1(X^2 + aY^2) + Y$ for some $X, Y \in \mathbb{F}$.*

In Theorem 1.3, $\mathrm{diag}(h_1, h_2, h_3)$ denotes the diagonal (3×3)-matrix whose entry in the i-th row and i-th column ($i \in \{1, 2, 3\}$) is equal to h_i.

The machinery that we will use to classify all $Sp(V, f)$-equivalence classes of trivectors belonging to $\mathcal{E}_{\mathbb{F}'}$ will be developed in Section 3. The vector space V can naturally be extended to a 6-dimensional vector space V' over $\mathbb{F}'$. We consider two forms f' and g on V' which will play an important role in the proof. The first form f' is just the alternating bilinear form on V' obtained by extending f. The second "form" g is usually not bilinear. At the end of Section 3 (Corollary 3.18), we will divide the family $\mathcal{E}_{\mathbb{F}'}$ of trivectors into three subfamilies such that trivectors belonging to distinct subfamilies are never $Sp(V, f)$-equivalent. We will show that these three subfamilies correspond to the three families of trivectors defined above ((E1'), (E2') and (E3')). Sections 4, 5 and 6 are devoted to the classification of the $Sp(V, f)$-equivalence classes of trivectors that belong to the three subfamilies described in Corollary 3.18.

We will see in Corollary 3.6 that the elements of $\mathcal{E}_{\mathbb{F}'}$ are trivectors of Type (D) when regarded as trivectors of V'. All $Sp(V', f')$-equivalence classes of trivectors of Type (D) of V' were determined in De Bruyn & Kwiatkowski [6]. We will recall this classification in Section 2. In Sections 4, 5 and 6, we will also determine those $Sp(V', f')$-equivalence classes of trivectors of Type (D) to which the elements of $\mathcal{E}_{\mathbb{F}'}$ belong.

2 On the trivectors of Type (D)

In this section, we recall the classification of the $Sp(V', f')$-equivalence classes of trivectors of Type (D) as given in De Bruyn & Kwiatkowski [6].

Let V' be a 6-dimensional vector space over the field $\mathbb{F}'$ equipped with a nondegenerate alternating bilinear form f'. Let $(\bar{e}^*_1, \bar{f}^*_1, \bar{e}^*_2, \bar{f}^*_2, \bar{e}^*_3, \bar{f}^*_3)$ be a fixed hyperbolic basis of (V', f'). We now define a number of trivectors of V'.

- We define

$$\gamma_1 \;:=\; \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*.$$

- For every $\lambda \in \mathbb{F}' \setminus \{0\}$, we define

$$\begin{aligned}\gamma_2(\lambda) &:= \lambda \cdot \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \bar{e}_2^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* + \bar{f}_1^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*,\\ \gamma_5(\lambda) &:= \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_2^* + \bar{f}_3^*) - \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*.\end{aligned}$$

- For all $\lambda_1, \lambda_2 \in \mathbb{F}' \setminus \{0\}$, we define

$$\begin{aligned}\gamma_3(\lambda_1,\lambda_2) &:= \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*,\\ \gamma_4(\lambda_1,\lambda_2) &:= \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_3^* + \lambda_1 \cdot \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \lambda_2 \cdot \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_2^*.\end{aligned}$$

- If $\mathsf{char}(\mathbb{F}') \neq 2$, then we define the following additional trivector:

$$\gamma_6 \;:=\; -\bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge \bar{f}_1^* + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*.$$

- If $|\mathbb{F}'| = 2$, then we define the following additional trivector:

$$\gamma_7 \;:=\; \bar{e}_1^* \wedge \bar{e}_2^* \wedge \bar{f}_2^* + \bar{e}_2^* \wedge \bar{e}_3^* \wedge (\bar{f}_1^* + \bar{f}_3^*) + \bar{e}_3^* \wedge \bar{e}_1^* \wedge \bar{f}_3^*.$$

Let $i \in \{1,2,\dots,7\}$. Any trivector of V' which is $Sp(V',f')$-equivalent with some γ_i-trivector defined above is called a *trivector of Type (Di)*. The following results were proved in De Bruyn & Kwiatkowski [6].

Proposition 2.1 ([6, Theorems 1.2, 1.3 and 1.4]) (1) *The trivectors of Type (D) of V' are precisely the trivectors of Type (D1), (D2), (D3), (D4), (D5), (D6) and (D7).*

(2) *Let $i,j \in \{1,2,...,7\}$ with $i \neq j$. Then no trivector of Type (Di) is $Sp(V',f')$-equivalent with a trivector of Type (Dj).*

(3) *If $\lambda, \lambda' \in \mathbb{F}' \setminus \{0\}$, then the trivectors $\gamma_2(\lambda)$ and $\gamma_2(\lambda')$ are $Sp(V',f')$-equivalent if and only if $\lambda = \lambda'$.*

(4) *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}' \setminus \{0\}$. Then the trivectors $\gamma_3(\lambda_1,\lambda_2)$ and $\gamma_3(\lambda_1',\lambda_2')$ are $Sp(V',f')$-equivalent if and only if the matrices* $\begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_1\lambda_2} \end{bmatrix}$ *and* $\begin{bmatrix} \frac{1}{\lambda_1'} & 0 & 0 \\ 0 & \frac{1}{\lambda_2'} & 0 \\ 0 & 0 & \frac{1}{\lambda_1'\lambda_2'} \end{bmatrix}$ *are congruent.*

(5) *Let $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{F}' \setminus \{0\}$. Then the trivectors $\gamma_4(\lambda_1,\lambda_2)$ and $\gamma_4(\lambda_1',\lambda_2')$ are $Sp(V',f')$-equivalent if and only if $\lambda_1 = \lambda_1'$ and there exist $X, Y \in \mathbb{F}'$ such that $Y^2 + \lambda_1 XY + \lambda_1 X^2 = \frac{\lambda_2'}{\lambda_2}$.*

(6) *Let $\lambda, \lambda' \in \mathbb{F}' \setminus \{0\}$. If $\mathsf{char}(\mathbb{F}') = 2$, then $\gamma_5(\lambda)$ and $\gamma_5(\lambda')$ are $Sp(V',f')$-equivalent if and only if $\frac{\lambda+\lambda'}{\lambda\lambda'}$ is of the form $X^2 + X$ for some $X \in \mathbb{F}'$. If $\mathsf{char}(\mathbb{F}') \neq 2$, then the trivectors $\gamma_5(\lambda)$ and $\gamma_5(\lambda')$ are always $Sp(V',f')$-equivalent.*

Remark. If $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6\}$ is a basis of V', then $\chi = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_4 + \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{v}_5 + \bar{v}_3 \wedge \bar{v}_1 \wedge \bar{v}_6$ is a trivector of Type (D) of V'. By Lemma 3.1 of [6], the 3-space $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ of V' is uniquely determined by χ. We call it the *base 3-space* of χ. In the above list of trivectors of Type (D), γ_1 and $\gamma_2(\lambda)$ are those trivectors whose base 3-space is not totally isotropic, while $\gamma_3(\lambda_1, \lambda_2)$, $\gamma_4(\lambda_1, \lambda_2)$, $\gamma_5(\lambda)$, γ_6 and γ_7 are those trivectors whose base 3-space is totally isotropic.

3 Development of the machinery for the classification

The aim of this section is to develop the machinery that we will use to obtain our desired classification results.

Let $\mathbb{F}$ be a field of characteristic 2, denote by $\overline{\mathbb{F}}$ a fixed algebraic closure of $\mathbb{F}$ and suppose $\mathbb{F}' \subseteq \overline{\mathbb{F}}$ is the nonseparable quadratic extension of $\mathbb{F}$ defined by the irreducible quadratic polynomial $X^2 + a \in \mathbb{F}[X]$. Let δ be the unique element of $\mathbb{F}' \setminus \mathbb{F}$ such that $\delta^2 = a$.

Let V' be a 6-dimensional vector space over $\mathbb{F}'$ equipped with a nondegenerate alternating bilinear form f'. Denote by $\{\bar{v}_1^*, \bar{v}_2^*, \bar{v}_3^*, \bar{v}_4^*, \bar{v}_5^*, \bar{v}_6^*\}$ a fixed basis of V' such that $f'(\bar{v}_i^*, \bar{v}_j^*) \in \mathbb{F}$ for all $i, j \in \{1, 2, \ldots, 6\}$ (e.g., take for $(\bar{v}_1^*, \bar{v}_2^*, \bar{v}_3^*, \bar{v}_4^*, \bar{v}_5^*, \bar{v}_6^*)$ an arbitrary hyperbolic basis of (V', f')). Let V denote the set of all $\mathbb{F}$-linear combinations of the elements of $\{\bar{v}_1^*, \bar{v}_2^*, \bar{v}_3^*, \bar{v}_4^*, \bar{v}_5^*, \bar{v}_6^*\}$ and let f denote the restriction of f' to V. Then V can be regarded in a natural way as a 6-dimensional vector space over $\mathbb{F}$, and f defines a nondegenerate alternating bilinear form on V.

The following two lemmas are special cases of a more general result, see e.g. De Bruyn [2, Section 4].

Lemma 3.1 *For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V', f'), let π_B denote the linear map from $\bigwedge^3 V'$ to V' defined by*

$$\pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \pi_B(\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \pi_B(\bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{o},$$

$$\pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \pi_B(\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \pi_B(\bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{o},$$

$$\pi_B(\bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \pi_B(\bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_1, \pi_B(\bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2) = \pi_B(\bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{f}_1,$$

$$\pi_B(\bar{e}_2 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{e}_2, \pi_B(\bar{f}_2 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3) = \bar{f}_2,$$

$$\pi_B(\bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{e}_3 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{e}_3, \pi_B(\bar{f}_3 \wedge \bar{e}_1 \wedge \bar{f}_1) = \pi_B(\bar{f}_3 \wedge \bar{e}_2 \wedge \bar{f}_2) = \bar{f}_3.$$

Then π_B is independent of the chosen hyperbolic basis B of (V', f').

Lemma 3.2 *For every hyperbolic basis $B = (\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V', f'), let π'_B denote the linear map from $\bigwedge^4 V'$ to $\bigwedge^2 V'$ defined by*

$$\pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3) = \bar{e}_2 \wedge \bar{e}_3,\ \pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3) = \bar{e}_2 \wedge \bar{f}_3,\ \pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3) = \bar{f}_2 \wedge \bar{e}_3,$$

$$\pi'_B(\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3) = \bar{f}_2 \wedge \bar{f}_3,\ \pi'_B(\bar{e}_2 \wedge \bar{f}_2 \wedge \bar{e}_1 \wedge \bar{e}_3) = \bar{e}_1 \wedge \bar{e}_3,\ \pi'_B(\bar{e}_2 \wedge \bar{f}_2 \wedge \bar{e}_1 \wedge \bar{f}_3) = \bar{e}_1 \wedge \bar{f}_3,$$

$$\pi'_B(\bar e_2\wedge\bar f_2\wedge\bar f_1\wedge\bar e_3)=\bar f_1\wedge\bar e_3,\ \pi'_B(\bar e_2\wedge\bar f_2\wedge\bar f_1\wedge\bar f_3)=\bar f_1\wedge\bar f_3,\ \pi'_B(\bar e_3\wedge\bar f_3\wedge\bar e_1\wedge\bar e_2)=\bar e_1\wedge\bar e_2,$$
$$\pi'_B(\bar e_3\wedge\bar f_3\wedge\bar e_1\wedge\bar f_2)=\bar e_1\wedge\bar f_2,\ \pi'_B(\bar e_3\wedge\bar f_3\wedge\bar f_1\wedge\bar e_2)=\bar f_1\wedge\bar e_2,\ \pi'_B(\bar e_3\wedge\bar f_3\wedge\bar f_1\wedge\bar f_2)=\bar f_1\wedge\bar f_2,$$
$$\pi'_B(\bar e_1\wedge\bar f_1\wedge\bar e_2\wedge\bar f_2)=\bar e_1\wedge\bar f_1+\bar e_2\wedge\bar f_2,\qquad \pi'_B(\bar e_1\wedge\bar f_1\wedge\bar e_3\wedge\bar f_3)=\bar e_1\wedge\bar f_1+\bar e_3\wedge\bar f_3,$$
$$\pi'_B(\bar e_2\wedge\bar f_2\wedge\bar e_3\wedge\bar f_3)=\bar e_2\wedge\bar f_2+\bar e_3\wedge\bar f_3.$$

Then π'_B is independent of the chosen hyperbolic basis B of (V',f').

Let $\pi:\bigwedge^3 V'\cup\bigwedge^4 V'\to V'\cup\bigwedge^2 V'$ be the map which sends α to $\pi_B(\alpha)$ if $\alpha\in\bigwedge^3 V'$ and to $\pi'_B(\alpha)$ if $\alpha\in\bigwedge^4 V'$. Here, B is some arbitrary hyperbolic basis of (V',f'). Observe that by Lemmas 3.1 and 3.2, the map π is an invariant, that means, is independent of the considered hyperbolic basis B of (V',f').

Let $Sp(V,f)\cong Sp_6(\mathbb{F})$ and $Sp(V',f')\cong Sp_6(\mathbb{F}')$ denote the symplectic groups associated with the respective pairs (V,f) and (V',f'). Every $\theta\in GL(V)$ may be naturally extended to an element of $GL(V')$ which we will also denote by θ. Following this convention, we thus have that $GL(V)\subset GL(V')$ and $Sp(V,f)\subset Sp(V',f')$. Recall that if $\chi_1,\chi_2\in\bigwedge^3 V'$ and G is one of the groups $GL(V'),GL(V),Sp(V',f'),Sp(V,f)$, then χ_1 and χ_2 are called *G-equivalent* if $\chi_2=\bigwedge^3(\theta)(\chi_1)$ for some $\theta\in G$.

If we put $\mu_{\mathbb{F}'}=a+1$ and $\lambda_{\mathbb{F}'}=\frac{a+1}{a}$, then $\mathbb{F}'\subseteq\overline{\mathbb{F}}$ is the quadratic extension of $\mathbb{F}$ defined by the quadratic polynomial $\mu_{\mathbb{F}'}X^2+(\mu_{\mathbb{F}'}\lambda_{\mathbb{F}'}+\mu_{\mathbb{F}'}+\lambda_{\mathbb{F}'})X+\lambda_{\mathbb{F}'}=(a+1)(X^2+\frac{1}{a})\in\mathbb{F}[X]$. So, we can put

$$\chi^*_{\mathbb{F}'}:=\frac{a+1}{a}\cdot\bar v_1^*\wedge\bar v_3^*\wedge\bar v_5^*+(a+1)\cdot\bar v_2^*\wedge\bar v_4^*\wedge\bar v_6^*+(\bar v_1^*+\bar v_2^*)\wedge(\bar v_3^*+\bar v_4^*)\wedge(\bar v_5^*+\bar v_6^*).$$

We now define a certain map $\phi:\bigwedge^3 V'\to\bigwedge^3 V$. If $\chi\in\bigwedge^3 V'$, then there exist unique $\chi_1,\chi_2\in\bigwedge^3 V$ such that $\chi=\chi_1+\delta\chi_2$, and we define

$$\phi(\chi):=\frac{1}{a}\cdot\chi_1+\chi_2.$$

With this definition, we have

$$\chi^*_{\mathbb{F}'}:=\phi\Big((\bar v_1^*+\delta\bar v_2^*)\wedge(\bar v_3^*+\delta\bar v_4^*)\wedge(\bar v_5^*+\delta\bar v_6^*)\Big).$$

Lemma 3.3 *If $\theta\in GL(V)$, then $\bigwedge^3(\theta)\circ\phi=\phi\circ\bigwedge^3(\theta)$.*

Proof. Let $\chi=\chi_1+\delta\chi_2$ be an arbitrary element of $\bigwedge^3 V'$, where $\chi_1,\chi_2\in\bigwedge^3 V$. Since $\theta\in GL(V)$, $\bigwedge^3(\theta)(\chi_1)\in\bigwedge^3 V$ and $\bigwedge^3(\theta)(\chi_2)\in\bigwedge^3 V$. Now, $\bigwedge^3(\theta)\circ\phi(\chi)=\bigwedge^3(\theta)(\frac{1}{a}\cdot\chi_1+\chi_2)=\frac{1}{a}\cdot\bigwedge^3(\theta)(\chi_1)+\bigwedge^3(\theta)(\chi_2)$ and $\phi\circ\bigwedge^3(\theta)(\chi)=\phi\Big(\bigwedge^3(\theta)(\chi_1)+\delta\cdot\bigwedge^3(\theta)(\chi_2)\Big)=\frac{1}{a}\cdot\bigwedge^3(\theta)(\chi_1)+\bigwedge^3(\theta)(\chi_2)$. ■

The following observation should be clear.

Lemma 3.4 *The trivectors of Type (E) of V belonging to $\mathcal{E}_{\mathbb{F}'}$ are precisely the trivectors of the form $\phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big)$, where $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ is some basis of V.*

The proof of the following lemma consists of a straightforward computation.

Lemma 3.5 *If $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_6$ are vectors of V, then $\phi\Big((\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)\Big) = (\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\frac{1}{a}\bar{v}_5 + \bar{v}_6) + (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6) \wedge (\frac{1}{a}\bar{v}_1 + \bar{v}_2) + (\bar{v}_5 + \delta\bar{v}_6) \wedge (\bar{v}_1 + \delta\bar{v}_2) \wedge (\frac{1}{a}\bar{v}_3 + \bar{v}_4)$.*

The following is a straightforward corollary of Lemma 3.5.

Corollary 3.6 *The trivectors of Type (E) of V belonging to $\mathcal{E}_{\mathbb{F}'}$ are trivectors of Type (D) when regarded as trivectors of V'.*

Lemma 3.7 *If $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ and $\{\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4, \bar{w}_5, \bar{w}_6\}$ are two bases of V such that $\phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big) = \phi\Big((\bar{w}_1 + \delta\bar{w}_2) \wedge (\bar{w}_3 + \delta\bar{w}_4) \wedge (\bar{w}_5 + \delta\bar{w}_6)\Big)$, then $(\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6) = (\bar{w}_1 + \delta\bar{w}_2) \wedge (\bar{w}_3 + \delta\bar{w}_4) \wedge (\bar{w}_5 + \delta\bar{w}_6)$.*

Proof. If $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5, \bar{u}_6\}$ and $\{\bar{u}'_1, \bar{u}'_2, \bar{u}'_3, \bar{u}'_4, \bar{u}'_5, \bar{u}'_6\}$ are two bases of V' such that $\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_4 + \bar{u}_2 \wedge \bar{u}_3 \wedge \bar{u}_5 + \bar{u}_3 \wedge \bar{u}_1 \wedge \bar{u}_6 = \bar{u}'_1 \wedge \bar{u}'_2 \wedge \bar{u}'_4 + \bar{u}'_2 \wedge \bar{u}'_3 \wedge \bar{u}'_5 + \bar{u}'_3 \wedge \bar{u}'_1 \wedge \bar{u}'_6$, then $\langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle = \langle \bar{u}'_1, \bar{u}'_2, \bar{u}'_3 \rangle$, see [6, Lemma 3.1]. This fact in combination with Lemma 3.5 yields that $\langle \bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6 \rangle = \langle \bar{w}_1 + \delta\bar{w}_2, \bar{w}_3 + \delta\bar{w}_4, \bar{w}_5 + \delta\bar{w}_6 \rangle$. So, there exist $\lambda_1, \lambda_2 \in \mathbb{F}$ with $(\lambda_1, \lambda_2) \neq (0, 0)$ such that $(\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6) = (\lambda_1+\lambda_2\delta)\cdot(\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6)$. Now, let χ_1 and χ_2 be the unique elements of $\bigwedge^3 V$ such that $(\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6) = \chi_1+\delta\chi_2$. Then χ_1 and χ_2 are linearly independent and $(\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6) = (\chi_1+\delta\chi_2)\cdot(\lambda_1+\delta\lambda_2) = (\lambda_1\cdot\chi_1+a\lambda_2\cdot\chi_2)+\delta(\lambda_1\cdot\chi_2+\lambda_2\cdot\chi_1)$. From $\phi\Big((\bar{w}_1 + \delta\bar{w}_2) \wedge (\bar{w}_3 + \delta\bar{w}_4) \wedge (\bar{w}_5 + \delta\bar{w}_6)\Big) = \phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big)$, we find

$$\frac{1}{a} \cdot \chi_1 + \chi_2 = \frac{\lambda_1}{a} \cdot \chi_1 + \lambda_2 \cdot \chi_2 + \lambda_1 \cdot \chi_2 + \lambda_2 \cdot \chi_1.$$

Since χ_1 and χ_2 are linearly independent, we find $\frac{1}{a} = \frac{\lambda_1}{a} + \lambda_2$ and $1 = \lambda_1 + \lambda_2$. Hence, $\lambda_1 = 1$, $\lambda_2 = 0$ and $(\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6) = (\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6)$. ■

Let Ω denote the set of all trivectors of V' of the form $(\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)$, where $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ is some basis of V. Then $\phi : \Omega \to \bigwedge^3 V$ defines a bijection between Ω and the set of trivectors of Type (E) of V belonging to $\mathcal{E}_{\mathbb{F}'}$.

Lemma 3.8 *Let $\chi_1, \chi_2 \in \Omega$. Then $\phi(\chi_1)$ and $\phi(\chi_2)$ are $Sp(V, f)$-equivalent if and only if χ_1 and χ_2 are $Sp(V, f)$-equivalent.*

Proof. The trivectors χ_1 and χ_2 are $Sp(V,f)$-equivalent if and only if $\chi_2 = \bigwedge^3(\theta)(\chi_1)$ for some $\theta \in Sp(V,f)$, i.e., if and only if $\phi(\chi_2) = \phi\Big(\bigwedge^3(\theta)(\chi_1)\Big) = \bigwedge^3(\theta)(\phi(\chi_1))$ for some $\theta \in Sp(V,f)$. The latter condition is equivalent with $\phi(\chi_1)$ and $\phi(\chi_2)$ being $Sp(V,f)$-equivalent. ■

The determination of the $Sp(V,f)$-equivalence classes of trivectors of Type (E) contained in $\mathcal{E}_{\mathbb{F}'}$ is thus equivalent with the determination of the $Sp(V,f)$-orbits on the elements of Ω.

For every vector $\bar{v} \in V'$, we define $\mathsf{Re}(\bar{v}) := \bar{v}_1$ and $\mathsf{Im}(\bar{v}) := \bar{v}_2$, where $\bar{v}_1$ and $\bar{v}_2$ are the unique vectors of V for which $\bar{v} = \bar{v}_1 + \delta\bar{v}_2$. For every $\eta \in \mathbb{F}'$, we define $\mathsf{Re}(\eta) := \eta_1$ and $\mathsf{Im}(\eta) := \eta_2$, where η_1 and η_2 are the unique elements of $\mathbb{F}$ for which $\eta = \eta_1 + \delta\eta_2$. If $A = (a_{ij})_{1\le i,j\le 3}$ is a (3×3)-matrix with entries in $\mathbb{F}'$, then $\mathsf{Re}(A)$ [resp., $\mathsf{Im}(A)$] denotes the (3×3)-matrix whose (i,j)-th entry is equal to $\mathsf{Re}(a_{ij})$ [resp., $\mathsf{Im}(a_{ij})$] $(i,j \in \{1,2,3\})$. If $*$ is a vector of V', an element of $\mathbb{F}'$ or a (3×3)-matrix over $\mathbb{F}'$, then $\mathsf{Re}(*)$ and $\mathsf{Im}(*)$ are respectively called the *real* and *imaginary part* of $*$.

Let $\tau : V' \to V'$ be the following map:

$$(\bar{v}_1 + \delta\bar{v}_2)^\tau := \bar{v}_2 + \delta\bar{v}_1 \qquad (\bar{v}_1, \bar{v}_2 \in V).$$

Clearly, $\tau \circ \theta = \theta \circ \tau$ for every $\theta \in GL(V)$, $(\bar{x}_1 + \bar{x}_2)^\tau = \bar{x}_1^\tau + \bar{x}_2^\tau$ for all $\bar{x}_1, \bar{x}_2 \in V'$ and $(\lambda\bar{x})^\tau = \lambda\bar{x}^\tau$ for every $\bar{x} \in V'$ and every $\lambda \in \mathbb{F}$.

Let $g : V' \times V' \to \mathbb{F}'$ be the following map:

$$g(\bar{x},\bar{y}) := f'(\bar{x},\bar{y}^\tau) \qquad (\bar{x},\bar{y} \in V').$$

In the following lemma, we collect some properties of the map g.

Lemma 3.9 (1) *If $\bar{x},\bar{y} \in V'$ and $\theta \in Sp(V,f)$, then $g(\bar{x}^\theta,\bar{y}^\theta) = g(\bar{x},\bar{y})$.*

(2) *If $\bar{x}_1,\bar{x}_2,\bar{y}_1,\bar{y}_2 \in V'$, then $g(\bar{x}_1+\bar{x}_2,\bar{y}_1) = g(\bar{x}_1,\bar{y}_1) + g(\bar{x}_2,\bar{y}_1)$ and $g(\bar{x}_1,\bar{y}_1+\bar{y}_2) = g(\bar{x}_1,\bar{y}_1) + g(\bar{x}_1,\bar{y}_2)$.*

(3) *If $\bar{x},\bar{y} \in V'$ and $\lambda \in \mathbb{F}'$, then $g(\lambda\bar{x},\bar{y}) = \lambda \cdot g(\bar{x},\bar{y})$ and $g(\bar{x},\lambda\bar{y}) = \lambda \cdot g(\bar{x},\bar{y}) + (a+1)\cdot \mathsf{Im}(\lambda)\cdot f'(\bar{x},\bar{y})$.*

Proof. (1) We have $g(\bar{x}^\theta,\bar{y}^\theta) = f'(\bar{x}^\theta,(\bar{y}^\theta)^\tau) = f'(\bar{x}^\theta,(\bar{y}^\tau)^\theta) = f'(\bar{x},\bar{y}^\tau) = g(\bar{x},\bar{y})$.

(2) This follows from the fact that $(\bar{y}_1+\bar{y}_2)^\tau = \bar{y}_1^\tau + \bar{y}_2^\tau$ and the fact that f' is bilinear.

(3) The first equality follows from the fact that f' is bilinear in its first component. To prove the second equality, put $\lambda := \lambda_1 + \delta\lambda_2$ and $\bar{y} := \bar{y}_1 + \delta\bar{y}_2$ where $\lambda_1,\lambda_2 \in \mathbb{F}$ and $\bar{y}_1,\bar{y}_2 \in V$. Then we have that

$$\begin{aligned}
g(\bar{x},\lambda\bar{y}) &= g(\bar{x},(\lambda_1+\delta\lambda_2)\cdot(\bar{y}_1+\delta\bar{y}_2)) \\
&= g(\bar{x},\lambda_1\bar{y}_1 + a\lambda_2\bar{y}_2 + \delta(\lambda_1\bar{y}_2+\lambda_2\bar{y}_1)) \\
&= f'(\bar{x},\lambda_1\bar{y}_2+\lambda_2\bar{y}_1+\delta(\lambda_1\bar{y}_1+a\lambda_2\bar{y}_2)) \\
&= \lambda_1\cdot f'(\bar{x},\bar{y}_2) + \lambda_2\cdot f'(\bar{x},\bar{y}_1) + \delta\cdot\Big(\lambda_1\cdot f'(\bar{x},\bar{y}_1) + \lambda_2 a\cdot f'(\bar{x},\bar{y}_2)\Big).
\end{aligned}$$

On the other hand, $\lambda \cdot g(\bar{x}, \bar{y}) + (a+1) \cdot \mathsf{Im}(\lambda) \cdot f'(\bar{x}, \bar{y})$ is equal to

$$(\lambda_1 + \lambda_2\delta) \cdot f'(\bar{x}, \bar{y}_2 + \delta\bar{y}_1) + (a+1)\lambda_2 \cdot f'(\bar{x}, \bar{y}_1 + \delta\bar{y}_2) =$$

$$\lambda_1 \cdot f'(\bar{x}, \bar{y}_2) + \lambda_1\delta \cdot f'(\bar{x}, \bar{y}_1) + \lambda_2\delta \cdot f'(\bar{x}, \bar{y}_2) + \lambda_2 a \cdot f'(\bar{x}, \bar{y}_1) + (a+1)\lambda_2 \cdot f'(\bar{x}, \bar{y}_1) + (a+1)\lambda_2\delta \cdot f'(\bar{x}, \bar{y}_2)$$

$$= \lambda_1 \cdot f'(\bar{x}, \bar{y}_2) + \lambda_1\delta \cdot f'(\bar{x}, \bar{y}_1) + a\lambda_2\delta \cdot f'(\bar{x}, \bar{y}_2) + \lambda_2 \cdot f'(\bar{x}, \bar{y}_1).$$

So $g(\bar{x}, \lambda\bar{y}) = \lambda \cdot g(\bar{x}, \bar{y}) + (a+1) \cdot \mathsf{Im}(\lambda) \cdot f'(\bar{x}, \bar{y})$. ∎

If $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k$ are $k \geq 1$ vectors of V' and $h \in \{f', g\}$, then $M_h(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k)$ denotes the $(k \times k)$-matrix over $\mathbb{F}'$ whose (i,j)-th entry is equal to $h(\bar{u}_i, \bar{u}_j)$ ($i, j \in \{1, 2, \ldots, k\}$).

Lemma 3.10 *Let $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ and $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ be two sets of linearly independent vectors of V' such that $\langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle = \langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$. Let $A = (a_{ij})_{1 \leq i,j \leq 3}$ be the (3×3)-matrix over $\mathbb{F}'$ such that $[\bar{v}_1, \bar{v}_2, \bar{v}_3]^{\mathrm{T}} = A \cdot [\bar{u}_1, \bar{u}_2, \bar{u}_3]^{\mathrm{T}}$. Then*

$$\begin{aligned} M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) &= A \cdot M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot A^{\mathrm{T}}, \\ M_g(\bar{v}_1, \bar{v}_2, \bar{v}_3) &= A \cdot M_g(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot A^{\mathrm{T}} + (a+1) \cdot A \cdot M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot \mathsf{Im}(A)^{\mathrm{T}}. \end{aligned}$$

Proof. For all $i, j \in \{1, 2, 3\}$, we have

$$f'(\bar{v}_i, \bar{v}_j) = f'(\sum_{k=1}^{3} a_{ik}\bar{u}_k, \sum_{l=1}^{3} a_{jl}\bar{u}_l) = \sum_{k=1}^{3}\sum_{l=1}^{3} a_{ik} \cdot f'(\bar{u}_k, \bar{u}_l) \cdot a_{jl} = \Big(A \cdot M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot A^{\mathrm{T}}\Big)_{ij}.$$

Invoking Lemma 3.9, we also have

$$\begin{aligned} g(\bar{v}_i, \bar{v}_j) &= g(\sum_{k=1}^{3} a_{ik}\bar{u}_k, \sum_{l=1}^{3} a_{jl}\bar{u}_l) = \sum_{k=1}^{3}\sum_{l=1}^{3} a_{ik} g(\bar{u}_k, a_{jl}\bar{u}_l) \\ &= \sum_{k=1}^{3}\sum_{l=1}^{3} \Big(a_{ik} \cdot g(\bar{u}_k, \bar{u}_l) \cdot a_{jl} + (a+1) \cdot a_{ik} \cdot f'(\bar{u}_k, \bar{u}_l) \cdot \mathsf{Im}(a_{jl})\Big) \\ &= \Big(A \cdot M_g(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot A^{\mathrm{T}} + (a+1) \cdot A \cdot M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot (\mathsf{Im}(A))^{\mathrm{T}}\Big)_{ij}. \end{aligned}$$

Hence, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = A \cdot M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot A^{\mathrm{T}}$ and $M_g(\bar{v}_1, \bar{v}_2, \bar{v}_3) = A \cdot M_g(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot A^{\mathrm{T}} + (a+1) \cdot A \cdot M_{f'}(\bar{u}_1, \bar{u}_2, \bar{u}_3) \cdot \mathsf{Im}(A)^{\mathrm{T}}$. ∎

Lemma 3.11 *Let $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ and $\{\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4, \bar{w}_5, \bar{w}_6\}$ be two bases of V, and let θ be the unique element of $GL(V)$ mapping $(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6)$ to $(\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4, \bar{w}_5, \bar{w}_6)$. Then $\theta \in Sp(V, f)$ if and only if $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = M_{f'}(\bar{w}_1 + \delta\bar{w}_2, \bar{w}_3 + \delta\bar{w}_4, \bar{w}_5 + \delta\bar{w}_6)$ and $M_g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = M_g(\bar{w}_1 + \delta\bar{w}_2, \bar{w}_3 + \delta\bar{w}_4, \bar{w}_5 + \delta\bar{w}_6)$.*

Proof. Suppose $\theta \in Sp(V, f)$. Then for all $i, j \in \{1, 3, 5\}$, we have $f'(\bar{w}_i + \delta\bar{w}_{i+1}, \bar{w}_j + \delta\bar{w}_{j+1}) = f'(\bar{v}_i^\theta + \delta\bar{v}_{i+1}^\theta, \bar{v}_j^\theta + \delta\bar{v}_{j+1}^\theta) = f'(\bar{v}_i + \delta\bar{v}_{i+1}, \bar{v}_j + \delta\bar{v}_{j+1})$ and $g(\bar{w}_i + \delta\bar{w}_{i+1}, \bar{w}_j + \delta\bar{w}_{j+1}) = g(\bar{v}_i^\theta + \delta\bar{v}_{i+1}^\theta, \bar{v}_j^\theta + \delta\bar{v}_{j+1}^\theta) = g(\bar{v}_i + \delta\bar{v}_{i+1}, \bar{v}_j + \delta\bar{v}_{j+1})$. It follows that $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 +$

$\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6) = M_{f'}(\bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6)$ and $M_g(\bar{v}_1+\delta\bar{v}_2, \bar{v}_3+\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6) = M_g(\bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6)$.

Conversely, suppose that $M_{f'}(\bar{v}_1+\delta\bar{v}_2, \bar{v}_3+\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6) = M_{f'}(\bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6)$ and $M_g(\bar{v}_1+\delta\bar{v}_2, \bar{v}_3+\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6) = M_g(\bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6)$. Let $i,j \in \{1,3,5\}$. From $f'(\bar{v}_i+\delta\bar{v}_{i+1}, \bar{v}_j+\delta\bar{v}_{j+1}) = f'(\bar{w}_i+\delta\bar{w}_{i+1}, \bar{w}_j+\delta\bar{w}_{j+1})$ and $f'(\bar{v}_i+\delta\bar{v}_{i+1}, \bar{v}_{j+1}+\delta\bar{v}_j) = g(\bar{v}_i+\delta\bar{v}_{i+1}, \bar{v}_j+\delta\bar{v}_{j+1}) = g(\bar{w}_i+\delta\bar{w}_{i+1}, \bar{w}_j+\delta\bar{w}_{j+1}) = f'(\bar{w}_i+\delta\bar{w}_{i+1}, \bar{w}_{j+1}+\delta\bar{w}_j)$, we find $f'(\bar{v}_i+\delta\bar{v}_{i+1}, \bar{v}_j) = f'(\bar{w}_i+\delta\bar{w}_{i+1}, \bar{w}_j)$ and $f'(\bar{v}_i+\delta\bar{v}_{i+1}, \bar{v}_{j+1}) = f'(\bar{w}_i+\delta\bar{w}_{i+1}, \bar{w}_{j+1})$. Since $f'(\bar{v}_i,\bar{v}_j)$, $f'(\bar{v}_i,\bar{v}_{j+1})$, $f'(\bar{v}_{i+1},\bar{v}_j)$, $f'(\bar{v}_{i+1},\bar{v}_{j+1})$, $f'(\bar{w}_i,\bar{w}_j)$, $f'(\bar{w}_i,\bar{w}_{j+1})$, $f'(\bar{w}_{i+1},\bar{w}_j)$ and $f'(\bar{w}_{i+1},\bar{w}_{j+1})$ belong to $\mathbb{F}$, the latter two equations imply that $f'(\bar{v}_i,\bar{v}_j) = f'(\bar{w}_i,\bar{w}_j)$, $f'(\bar{v}_i,\bar{v}_{j+1}) = f'(\bar{w}_i,\bar{w}_{j+1})$, $f'(\bar{v}_{i+1},\bar{v}_j) = f'(\bar{w}_{i+1},\bar{w}_j)$ and $f'(\bar{v}_{i+1},\bar{v}_{j+1}) = f'(\bar{w}_{i+1},\bar{w}_{j+1})$. So, we have that $f'(\bar{v}_k^\theta, \bar{v}_l^\theta) = f'(\bar{v}_k,\bar{v}_l)$ for all $k,l \in \{1,2,3,4,5,6\}$. This implies that $\theta \in Sp(V,f)$. ∎

Lemma 3.12 *Let* $\{\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5,\bar{v}_6\}$ *and* $\{\bar{w}_1,\bar{w}_2,\bar{w}_3,\bar{w}_4,\bar{w}_5,\bar{w}_6\}$ *be two bases of* V. *Put* $\chi_1 := \phi\Big((\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)\Big)$, $\chi_2 := \phi\Big((\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6)\Big)$, $M_1 := M_{f'}(\bar{v}_1+\delta\bar{v}_2, \bar{v}_3+\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6)$, $M_2 := M_{f'}(\bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6)$, $N_1 := M_g(\bar{v}_1+\delta\bar{v}_2, \bar{v}_3+\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6)$ *and* $N_2 := M_g(\bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6)$. *Then* χ_1 *and* χ_2 *are* $Sp(V,f)$*-equivalent if and only if there exists a matrix* $A \in SL(3,\mathbb{F}')$ *such that* $M_2 = AM_1A^{\mathrm{T}}$ *and* $N_2 = AN_1A^{\mathrm{T}} + (a+1)\cdot A\cdot M_1\cdot \mathsf{Im}(A)^{\mathrm{T}}$.

Proof. Suppose χ_1 and χ_2 are $Sp(V,f)$-equivalent and let $\theta \in Sp(V,f)$ such that $\bigwedge^3(\theta)(\chi_1) = \chi_2$. Then $\phi\Big((\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6)\Big) = \bigwedge^3(\theta)\Big[\phi\Big((\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)\Big)\Big] = \phi\Big((\bar{v}_1^\theta+\delta\bar{v}_2^\theta)\wedge(\bar{v}_3^\theta+\delta\bar{v}_4^\theta)\wedge(\bar{v}_5^\theta+\delta\bar{v}_6^\theta)\Big)$. Hence, $\langle \bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6\rangle = \langle \bar{v}_1^\theta+\delta\bar{v}_2^\theta, \bar{v}_3^\theta+\delta\bar{v}_4^\theta, \bar{v}_5^\theta+\delta\bar{v}_6^\theta\rangle$ and $(\bar{v}_1^\theta+\delta\bar{v}_2^\theta)\wedge(\bar{v}_3^\theta+\delta\bar{v}_4^\theta)\wedge(\bar{v}_5^\theta+\delta\bar{v}_6^\theta) = (\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6)$ by Lemma 3.7. Now, let A be the nonsingular (3×3)-matrix over $\mathbb{F}'$ such that $[\bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6]^{\mathrm{T}} = A\cdot[\bar{v}_1^\theta+\delta\bar{v}_2^\theta, \bar{v}_3^\theta+\delta\bar{v}_4^\theta, \bar{v}_5^\theta+\delta\bar{v}_6^\theta]^{\mathrm{T}}$. Since $(\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6) = (\bar{v}_1^\theta+\delta\bar{v}_2^\theta)\wedge(\bar{v}_3^\theta+\delta\bar{v}_4^\theta)\wedge(\bar{v}_5^\theta+\delta\bar{v}_6^\theta)$, we have $det(A) = 1$. Now, $M_{f'}(\bar{v}_1^\theta+\delta\bar{v}_2^\theta, \bar{v}_3^\theta+\delta\bar{v}_4^\theta, \bar{v}_5^\theta+\delta\bar{v}_6^\theta) = M_1$ and $M_g(\bar{v}_1^\theta+\delta\bar{v}_2^\theta, \bar{v}_3^\theta+\delta\bar{v}_4^\theta, \bar{v}_5^\theta+\delta\bar{v}_6^\theta) = N_1$. Lemma 3.10 now implies that $M_2 = AM_1A^{\mathrm{T}}$ and $N_2 = AN_1A^{\mathrm{T}} + (a+1)\cdot A\cdot M_1\cdot\mathsf{Im}(A)^{\mathrm{T}}$.

Conversely, suppose that $M_2 = AM_1A^{\mathrm{T}}$ and $N_2 = AN_1A^{\mathrm{T}} + (a+1)\cdot A\cdot M_1\cdot\mathsf{Im}(A)^{\mathrm{T}}$ for some matrix $A \in SL(3,\mathbb{F}')$. Let $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, $\bar{u}_4$, $\bar{u}_5$ and $\bar{u}_6$ be the unique vectors of V such that $[\bar{w}_1+\delta\bar{w}_2, \bar{w}_3+\delta\bar{w}_4, \bar{w}_5+\delta\bar{w}_6]^{\mathrm{T}} = A\cdot[\bar{u}_1+\delta\bar{u}_2, \bar{u}_3+\delta\bar{u}_4, \bar{u}_5+\delta\bar{u}_6]^{\mathrm{T}}$. Since $\det(A) = 1$, we have $(\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6) = (\bar{u}_1+\delta\bar{u}_2)\wedge(\bar{u}_3+\delta\bar{u}_4)\wedge(\bar{u}_5+\delta\bar{u}_6)$ and hence $\chi_2 = \phi\Big((\bar{u}_1+\delta\bar{u}_2)\wedge(\bar{u}_3+\delta\bar{u}_4)\wedge(\bar{u}_5+\delta\bar{u}_6)\Big)$. From Lemma 3.10, we easily derive that $M_{f'}(\bar{u}_1+\delta\bar{u}_2, \bar{u}_3+\delta\bar{u}_4, \bar{u}_5+\delta\bar{u}_6) = M_1$ and $M_g(\bar{u}_1+\delta\bar{u}_2, \bar{u}_3+\delta\bar{u}_4, \bar{u}_5+\delta\bar{u}_6) = N_1$. Since also $M_{f'}(\bar{v}_1+\delta\bar{v}_2, \bar{v}_3+\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6) = M_1$ and $M_g(\bar{v}_1+\delta\bar{v}_2, \bar{v}_3+\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6) = N_1$, by Lemma 3.11 there exists a $\theta \in Sp(V,f)$ mapping $(\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5,\bar{v}_6)$ to $(\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5,\bar{u}_6)$. Then $\chi_2 = \phi\Big((\bar{u}_1+\delta\bar{u}_2)\wedge(\bar{u}_3+\delta\bar{u}_4)\wedge(\bar{u}_5+\delta\bar{u}_6)\Big) = \phi\Big((\bar{v}_1^\theta+\delta\bar{v}_2^\theta)\wedge(\bar{v}_3^\theta+\delta\bar{v}_4^\theta)\wedge(\bar{v}_5^\theta+\delta\bar{v}_6^\theta)\Big) = \bigwedge^3(\theta)\Big[\phi\Big((\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)\Big)\Big] = \bigwedge^3(\theta)(\chi_1)$. So, χ_1 and χ_2 are $Sp(V,f)$-equivalent. ∎

Lemma 3.13 (1) *For all $\bar{x} \in V'$, we have $g(\bar{x}, \bar{x}) \in \mathbb{F}$.*

(2) *For all $\bar{x}, \bar{y} \in V'$, we have $g(\bar{x}, \bar{y}) + g(\bar{y}, \bar{x}) = (a+1) \cdot \mathsf{Im}(f'(\bar{x}, \bar{y}))$.*

(3) *For all $\bar{x}, \bar{y} \in V'$, we have $f'(\bar{x}^\tau, \bar{y}^\tau) + f'(\bar{x}, \bar{y}) = (a+1) \cdot \mathsf{Im}(g(\bar{x}, \bar{y}))$.*

Proof. (1) If $\bar{v}_1$ and $\bar{v}_2$ are the unique vectors of V such that $\bar{x} = \bar{v}_1 + \delta \bar{v}_2$, then $g(\bar{x}, \bar{x}) = f'(\bar{v}_1 + \delta \bar{v}_2, \bar{v}_2 + \delta \bar{v}_1) = (1 + \delta^2) \cdot f'(\bar{v}_1, \bar{v}_2) = (a+1) \cdot f'(\bar{v}_1, \bar{v}_2) \in \mathbb{F}$.

(2) Let $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$ and $\bar{v}_4$ be the unique vectors of V such that $\bar{x} = \bar{v}_1 + \delta \bar{v}_2$ and $\bar{y} = \bar{v}_3 + \delta \bar{v}_4$. Then $g(\bar{x}, \bar{y}) + g(\bar{y}, \bar{x}) = f'(\bar{v}_1 + \delta \bar{v}_2, \bar{v}_4 + \delta \bar{v}_3) + f'(\bar{v}_3 + \delta \bar{v}_4, \bar{v}_2 + \delta \bar{v}_1) = f'(\bar{v}_1, \bar{v}_4) + a \cdot f'(\bar{v}_2, \bar{v}_3) + \delta \cdot f'(\bar{v}_2, \bar{v}_4) + \delta \cdot f'(\bar{v}_1, \bar{v}_3) + f'(\bar{v}_3, \bar{v}_2) + \delta \cdot f'(\bar{v}_3, \bar{v}_1) + \delta \cdot f'(\bar{v}_4, \bar{v}_2) + a \cdot f'(\bar{v}_4, \bar{v}_1) = (a+1) \cdot (f'(\bar{v}_1, \bar{v}_4) + f'(\bar{v}_2, \bar{v}_3))$ and $\mathsf{Im}(f'(\bar{v}_1 + \delta \bar{v}_2, \bar{v}_3 + \delta \bar{v}_4)) = f'(\bar{v}_2, \bar{v}_3) + f'(\bar{v}_1, \bar{v}_4)$.

(3) By (2), we have $f'(\bar{x}^\tau, \bar{y}^\tau) + f'(\bar{x}, \bar{y}) = g(\bar{y}^\tau, \bar{x}) + g(\bar{x}, \bar{y}^\tau) = (a+1) \cdot \mathsf{Im}(f'(\bar{x}, \bar{y}^\tau)) = (a+1) \cdot \mathsf{Im}(g(\bar{x}, \bar{y}))$. ∎

The following is an immediate consequence of Lemma 3.13.

Corollary 3.14 *Suppose $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6 \in V$. Then $M_{f'}(\bar{v}_1 + \delta \bar{v}_2, \bar{v}_3 + \delta \bar{v}_4, \bar{v}_5 + \delta \bar{v}_6, \bar{v}_2 + \delta \bar{v}_1, \bar{v}_4 + \delta \bar{v}_3, \bar{v}_6 + \delta \bar{v}_5)$ is equal to*

$$\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

where $M_1 = M_{f'}(\bar{v}_1 + \delta \bar{v}_2, \bar{v}_3 + \delta \bar{v}_4, \bar{v}_5 + \delta \bar{v}_6)$, $M_2 = M_g(\bar{v}_1 + \delta \bar{v}_2, \bar{v}_3 + \delta \bar{v}_4, \bar{v}_5 + \delta \bar{v}_6)$, $M_4 = M_1 + (a+1) \cdot \mathsf{Im}(M_2)$ and $M_3 = M_2^{\mathrm{T}} = M_2 + (a+1) \cdot \mathsf{Im}(M_1)$. So, if all entries of M_1 belong to $\mathbb{F}$, then $M_2 = M_3$ is a symmetric matrix.

Let O_3 denote the (3×3)-matrix over $\mathbb{F}'$ all whose entries are equal to 0. Let M^* denote the following (3×3)-matrix over $\mathbb{F}'$:

$$M^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

For a proof of the following lemma, see e.g. [5, Lemma 3.14].

Lemma 3.15 ([5]) *Let $A = (a_{ij})_{1 \leq i,j \leq 3}$ be a matrix of $SL(3, \mathbb{F}')$. Then $A \cdot M^* \cdot A^{\mathrm{T}} = M^*$ if and only if $a_{11} = 1$, $a_{12} = a_{13} = 0$ and $a_{22}a_{33} - a_{23}a_{32} = 1$.*

Lemma 3.16 *Let $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$ be three linearly independent vectors of V'. Then there exist three linearly independent vectors $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$ and $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ is equal to either O_3 or M^*.*

Proof. Suppose $\langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle$ is a totally isotropic 3-dimensional subspace. Then put $(\bar{v}_1, \bar{v}_2, \bar{v}_3) := (\bar{u}_1, \bar{u}_2, \bar{u}_3)$. Clearly, $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$ and $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = O_3$.

Suppose $\langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle$ is not totally isotropic. Then there exist $\bar{v}_2, \bar{v}_3 \in \langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle$ such that $f'(\bar{v}_2, \bar{v}_3) = 1$. Let U denote the unique 1-space of $\langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle$ such that $f'(\bar{u}, \bar{v}) = 0$ for every $\bar{u} \in U$ and every $\bar{v} \in \langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle$. If $\bar{v}_1$ denotes the unique vector of U such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3$, then $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3) = M^*$. ∎

Lemma 3.17 *Let* $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5, \bar{u}_6\}$, $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ *and* $\{\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4, \bar{w}_5, \bar{w}_6\}$ *be three bases of* V. *Suppose* $M_{f'}(\bar{u}_1 + \delta\bar{u}_2, \bar{u}_3 + \delta\bar{u}_4, \bar{u}_5 + \delta\bar{u}_6) = O_3$, $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = M_{f'}(\bar{w}_1 + \delta\bar{w}_2, \bar{w}_3 + \delta\bar{w}_4, \bar{w}_5 + \delta\bar{w}_6) = M^*$, $g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_1 + \delta\bar{v}_2) \neq 0$ *and* $g(\bar{w}_1+\delta\bar{w}_2, \bar{w}_1+\delta\bar{w}_2) = 0$. *Then the trivectors* $\chi_1 := \phi\Big((\bar{u}_1+\delta\bar{u}_2)\wedge(\bar{u}_3+\delta\bar{u}_4)\wedge(\bar{u}_5+\delta\bar{u}_6)\Big)$, $\chi_2 := \phi\Big((\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)\Big)$ *and* $\chi_3 := \phi\Big((\bar{w}_1+\delta\bar{w}_2)\wedge(\bar{w}_3+\delta\bar{w}_4)\wedge(\bar{w}_5+\delta\bar{w}_6)\Big)$ *of* V *are mutually non-*$Sp(V,f)$*-equivalent.*

Proof. Suppose χ_1 and χ_i are $Sp(V,f)$-equivalent for some $i \in \{2,3\}$. Then by Lemma 3.12, there exists a matrix $A \in SL(3,\mathbb{F}')$ such that $M^* = A \cdot O_3 \cdot A^{\mathrm{T}} = O_3$, clearly a contradiction.

Suppose χ_2 and χ_3 are $Sp(V,f)$-equivalent. Then there exists a $\theta \in Sp(V,f)$ such that $\chi_3 = \bigwedge^3(\theta)(\chi_2)$. By Lemmas 3.3 and 3.7, this implies that $(\bar{w}_1 + \delta\bar{w}_2) \wedge (\bar{w}_3 + \delta\bar{w}_4) \wedge (\bar{w}_5 + \delta\bar{w}_6) = (\bar{v}_1^\theta + \delta\bar{v}_2^\theta) \wedge (\bar{v}_3^\theta + \delta\bar{v}_4^\theta) \wedge (\bar{v}_5^\theta + \delta\bar{v}_6^\theta)$. Since $f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}) = 0$ for all $\bar{v} \in \langle \bar{v}_1+\delta\bar{v}_2, \bar{v}_3+\delta\bar{v}_4, \bar{v}_5+\delta\bar{v}_6\rangle$, we have $f'(\bar{v}_1^\theta+\delta\bar{v}_2^\theta, \bar{w}) = 0$ for all $\bar{w} \in \langle \bar{v}_1^\theta+\delta\bar{v}_2^\theta, \bar{v}_3^\theta+\delta\bar{v}_4^\theta, \bar{v}_5^\theta+\delta\bar{v}_6^\theta\rangle = \langle \bar{w}_1 + \delta\bar{w}_2, \bar{w}_3 + \delta\bar{w}_4, \bar{w}_5 + \delta\bar{w}_6\rangle$. Since $M_{f'}(\bar{w}_1 + \delta\bar{w}_2, \bar{w}_3 + \delta\bar{w}_4, \bar{w}_5 + \delta\bar{w}_6) = M^*$, we have $\bar{v}_1^\theta + \delta\bar{v}_2^\theta = \eta(\bar{w}_1 + \delta\bar{w}_2)$ for some $\eta \in \mathbb{F}^*$. By (1) and (3) of Lemma 3.9, we then have $g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_1 + \delta\bar{v}_2) = g(\bar{v}_1^\theta + \delta\bar{v}_2^\theta, \bar{v}_1^\theta + \delta\bar{v}_2^\theta) = g(\eta(\bar{w}_1 + \delta\bar{w}_2), \eta(\bar{w}_1 + \delta\bar{w}_2)) = \eta^2 \cdot g(\bar{w}_1 + \delta\bar{w}_2, \bar{w}_1 + \delta\bar{w}_2)$. This is impossible since $g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_1 + \delta\bar{v}_2) \neq 0$ while $g(\bar{w}_1 + \delta\bar{w}_2, \bar{w}_1 + \delta\bar{w}_2) = 0$. ∎

The following is an immediate consequence of Lemmas 3.4, 3.16 and 3.17.

Corollary 3.18 *Precisely one of the following cases occurs for a trivector* $\chi \in \mathcal{E}_{\mathbb{F}'}$:

(*A*) $\chi = \phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big)$ *for some basis* $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ *of* V *satisfying* $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = O_3$;

(*B*) $\chi = \phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big)$ *for some basis* $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ *of* V *satisfying* $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = M^*$ *and* $g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_1 + \delta\bar{v}_2) \neq 0$;

(*C*) $\chi = \phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big)$ *for some basis* $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ *of* V *satisfying* $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = M^*$ *and* $g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_1 + \delta\bar{v}_2) = 0$.

So, we have three cases to consider when classifying all $Sp(V,f)$-equivalence classes of trivectors of Type (E) of V that are contained in $\mathcal{E}_{\mathbb{F}'}$. We will deal with each of these three cases in a separate section.

4 Treatment of Case (A) of Corollary 3.18

In this section, we determine the $Sp(V,f)$-equivalence classes of trivectors that are contained in the subfamily of $\mathcal{E}_{\mathbb{F}'}$ corresponding to Case (A) of Corollary 3.18.

Suppose χ is a trivector of Type (E) of V which is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ such that $\chi = \phi\Big((\bar{u}_1 + \delta\bar{u}_2) \wedge (\bar{u}_3 + \delta\bar{u}_4) \wedge (\bar{u}_5 + \delta\bar{u}_6)\Big)$ for some basis $\{\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4, \bar{u}_5, \bar{u}_6\}$ of V satisfying $M_{f'}(\bar{u}_1 + \delta\bar{u}_2, \bar{u}_3 + \delta\bar{u}_4, \bar{u}_5 + \delta\bar{u}_6) = O_3$.

Lemma 4.1 *Let $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ be a basis of V such that $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = O_3$. Then there exists a basis $\{\bar{w}_1, \bar{w}_2, \bar{w}_3\}$ of $W := \langle \bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6\rangle$ such that $\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 = (\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)$, $M_{f'}(\bar{w}_1, \bar{w}_2, \bar{w}_3) = O_3$ and $M_g(\bar{w}_1, \bar{w}_2, \bar{w}_3)$ is diagonal.*

Proof. Suppose $\{\bar{w}_1, \bar{w}_2, \bar{w}_3\}$ is a basis of W. Then $M_{f'}(\bar{w}_1, \bar{w}_2, \bar{w}_3) = O_3$. Corollary 3.14 implies that $M_g(\bar{w}_1, \bar{w}_2, \bar{w}_3)$ is symmetric. Also, since $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6, \bar{v}_2 + \delta\bar{v}_1, \bar{v}_4 + \delta\bar{v}_3, \bar{v}_6 + \delta\bar{v}_5)$ is nonsingular and $M_{f'}(\bar{w}_1, \bar{w}_2, \bar{w}_3) = O_3$, the matrix $M_g(\bar{w}_1, \bar{w}_2, \bar{w}_3)$ should be nonsingular. Lemma 3.9 now implies that g defines a nonsingular bilinear form on W. For every vector $\bar{w} \in W$, we denote by $\bar{w}^{\perp_g}$ the set of all vectors $\bar{w}' \in W$ for which $g(\bar{w}, \bar{w}') = 0$.

If all diagonal elements of $M_g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6)$ were equal to 0, then since $M_g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6)$ is (skew-)symmetric, it would also be singular which is impossible. So, there exist a $\bar{w}_1' \in \{\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6\}$ and $\bar{w}_2', \bar{w}_3' \in (\bar{w}_1')^{\perp_g}$ such that $g(\bar{w}_1', \bar{w}_1') \neq 0$ and $\bar{w}_1' \wedge \bar{w}_2' \wedge \bar{w}_3' = (\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)$. If $g(\bar{w}_2', \bar{w}_2') \neq 0$ or $g(\bar{w}_3', \bar{w}_3') \neq 0$, then there exist a $\bar{w}_2 \in \{\bar{w}_2', \bar{w}_3'\}$ and a $\bar{w}_3 \in \bar{w}_2^{\perp_g}$ such that $g(\bar{w}_2, \bar{w}_2) \neq 0$ and $\bar{w}_2 \wedge \bar{w}_3 = \bar{w}_2' \wedge \bar{w}_3'$. If we moreover put $\bar{w}_1 := \bar{w}_1'$, then $\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 = (\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)$ and $M_g(\bar{w}_1, \bar{w}_2, \bar{w}_3)$ is diagonal.

So, we may suppose that $g(\bar{w}_2', \bar{w}_2') = 0 = g(\bar{w}_3', \bar{w}_3')$. Then

$$M_g(\bar{w}_1', \bar{w}_2', \bar{w}_3') = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 \\ 0 & \mu_2 & 0 \end{bmatrix}$$

for some $\mu_1 \in \mathbb{F}^*$ and some $\mu_2 \in \mathbb{F}' \setminus \{0\}$. If we define $(\bar{w}_1, \bar{w}_2, \bar{w}_3) := (\bar{w}_1' + \frac{\mu_1}{\mu_2}\bar{w}_2' + \bar{w}_3', \bar{w}_2' + \frac{\mu_2}{\mu_1}\bar{w}_1', \bar{w}_3' + \bar{w}_1')$, then $\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 = \bar{w}_1' \wedge \bar{w}_2' \wedge \bar{w}_3' = (\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)$ and $M_g(\bar{w}_1, \bar{w}_2, \bar{w}_3) = \mathrm{diag}(\mu_1, \frac{\mu_2^2}{\mu_1}, \mu_1)$. ∎

By Lemma 4.1, we know that there exists a basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ of V such that $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = O_3$, $M_g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = \mathrm{diag}(h_1(a+1), h_2(a+1), h_3(a+1))$ and $\chi = \phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big)$ for some $h_1, h_2, h_3 \in \mathbb{F}^*$.

From $(a+1)h_i = g(\bar{v}_{2i-1} + \delta\bar{v}_{2i}, \bar{v}_{2i-1} + \delta\bar{v}_{2i}) = f'(\bar{v}_{2i-1} + \delta\bar{v}_{2i}, \bar{v}_{2i} + \delta\bar{v}_{2i-1}) = (a+1) \cdot f'(\bar{v}_{2i-1}, \bar{v}_{2i})$ for every $i \in \{1, 2, 3\}$, we find $f'(\bar{v}_1, \bar{v}_2) = h_1$, $f'(\bar{v}_3, \bar{v}_4) = h_2$ and $f'(\bar{v}_5, \bar{v}_6) = h_3$.

From $0 = g(\bar{v}_{2i-1} + \delta\bar{v}_{2i}, \bar{v}_{2j-1} + \delta\bar{v}_{2j}) = f'(\bar{v}_{2i-1} + \delta\bar{v}_{2i}, \bar{v}_{2j} + \delta\bar{v}_{2j-1}) = 0$ and $f'(\bar{v}_{2i-1} + \delta\bar{v}_{2i}, \bar{v}_{2j-1} + \delta\bar{v}_{2j}) = 0$, we find $f'(\bar{v}_{2i-1} + \delta\bar{v}_{2i}, \bar{v}_{2j}) = f'(\bar{v}_{2i-1} + \delta\bar{v}_{2i}, \bar{v}_{2j-1}) = 0$ and hence $f'(\bar{v}_{2i-1}, \bar{v}_{2j}) = f'(\bar{v}_{2i}, \bar{v}_{2j}) = f'(\bar{v}_{2i-1}, \bar{v}_{2j-1}) = f'(\bar{v}_{2i}, \bar{v}_{2j-1}) = 0$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. So, $f'(\bar{v}_i, \bar{v}_j) = 0$ if $i, j \in \{1, 2, 3, 4, 5, 6\}$ with $i \neq j$ and $\{i, j\}$ different from $\{1, 2\}$,

$\{3,4\}$ and $\{5,6\}$. This implies that there exists a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) such that

$$\bar{v}_1 = \bar{e}_1,\ \bar{v}_2 = h_1\bar{f}_1,\ \bar{v}_3 = \bar{e}_2,\ \bar{v}_4 = h_2\bar{f}_2,\ \bar{v}_5 = \bar{e}_3,\ \bar{v}_6 = h_3\bar{f}_3.$$

So,

$$\begin{aligned}
\chi &= \phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big) \\
&= \frac{a+1}{a} \cdot \bar{v}_1 \wedge \bar{v}_3 \wedge \bar{v}_5 + (a+1) \cdot \bar{v}_2 \wedge \bar{v}_4 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_2) \wedge (\bar{v}_3 + \bar{v}_4) \wedge (\bar{v}_5 + \bar{v}_6) \\
&= \frac{a+1}{a} \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + (a+1)h_1h_2h_3 \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 \\
&\quad +(\bar{e}_1 + h_1\bar{f}_1) \wedge (\bar{e}_2 + h_2\bar{f}_2) \wedge (\bar{e}_3 + h_3\bar{f}_3).
\end{aligned}$$

It follows that χ is $Sp(V,f)$-equivalent with $\chi_1(h_1,h_2,h_3)$.

Reversing the above procedure, we see that the trivector $\chi_1(h_1,h_2,h_3)$ is of the form $\phi\Big((\bar{v}_1^* + \delta\bar{v}_2^*) \wedge (\bar{v}_3^* + \delta\bar{v}_4^*) \wedge (\bar{v}_5^* + \delta\bar{v}_6^*)\Big)$, where $\{\bar{v}_1^*, \bar{v}_2^*, \bar{v}_3^*, \bar{v}_4^*, \bar{v}_5^*, \bar{v}_6^*\}$ is some basis of V satisfying $M_{f'}(\bar{v}_1^* + \delta\bar{v}_2^*, \bar{v}_3^* + \delta\bar{v}_4^*, \bar{v}_5^* + \delta\bar{v}_6^*) = O_3$ and $M_g(\bar{v}_1^* + \delta\bar{v}_2^*, \bar{v}_3^* + \delta\bar{v}_4^*, \bar{v}_5^* + \delta\bar{v}_6^*) = \mathrm{diag}(h_1(a+1), h_2(a+1), h_3(a+1))$. So, $\chi_1(h_1,h_2,h_3)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ by Lemma 3.4.

The following proposition, which is precisely Theorem 1.3(2), is a corollary of Lemma 3.12 and the above discussion.

Proposition 4.2 *Let $h_1, h_2, h_3, h_1', h_2', h_3' \in \mathbb{F}^*$. Then the two trivectors $\chi_1(h_1,h_2,h_3)$ and $\chi_1(h_1',h_2',h_3')$ of V are $Sp(V,f)$-equivalent if and only if there exists a matrix $A \in SL(3,\mathbb{F}')$ such that* $\mathrm{diag}(h_1',h_2',h_3') = A \cdot \mathrm{diag}(h_1,h_2,h_3) \cdot A^{\mathrm{T}}$.

By Corollary 3.6, we know that the trivector $\chi_1(h_1,h_2,h_3)$ is a trivector of Type (D) when regarded as a trivector of V'. One can now ask to which of the trivectors mentioned in Section 2 $\chi_1(h_1,h_2,h_3)$ is $Sp(V',f')$-equivalent to. The following proposition answers this question.

Proposition 4.3 *For all $h_1,h_2,h_3 \in \mathbb{F}^*$, the trivector $\chi_1(h_1,h_2,h_3)$ of V' is $Sp(V',f')$-equivalent with the trivector $\gamma_3(\frac{h_1}{h_3}, \frac{h_2}{h_3})$ of V'.*

Proof. The trivector $\chi_1(h_1,h_2,h_3) = (\bar{e}_1^* + \delta h_1\bar{f}_1^*) \wedge (\bar{e}_2^* + \delta h_2\bar{f}_2^*) \wedge (\frac{1}{a}\bar{e}_3^* + h_3\bar{f}_3^*) + (\bar{e}_2^* + \delta h_2\bar{f}_2^*) \wedge (\bar{e}_3^* + \delta h_3\bar{f}_3^*) \wedge (\frac{1}{a}\bar{e}_1^* + h_1\bar{f}_1^*) + (\bar{e}_3^* + \delta h_3\bar{f}_3^*) \wedge (\bar{e}_1^* + \delta h_1\bar{f}_1^*) \wedge (\frac{1}{a}\bar{e}_2^* + h_2\bar{f}_2^*)$ is equal to $\bar{e}_1' \wedge \bar{e}_2' \wedge \bar{f}_3' + \frac{h_1}{h_3} \cdot \bar{e}_2' \wedge \bar{e}_3' \wedge \bar{f}_1' + \frac{h_2}{h_3} \cdot \bar{e}_3' \wedge \bar{e}_1' \wedge \bar{f}_2'$, where $(\bar{e}_1', \bar{f}_1', \bar{e}_2', \bar{f}_2', \bar{e}_3', \bar{f}_3')$ is the hyperbolic basis of (V',f') defined by

$$\bar{e}_1' = \frac{(1+\delta)h_3}{\delta}(\bar{e}_1^* + \delta h_1\bar{f}_1^*),\ \bar{e}_2' = \frac{(1+\delta)h_3}{\delta}(\bar{e}_2^* + \delta h_2\bar{f}_2^*),\ \bar{e}_3' = \frac{(1+\delta)h_3}{\delta}(\bar{e}_3^* + \delta h_3\bar{f}_3^*),$$

$$\bar{f}_1' = \frac{a}{(a+1)h_1h_3}(\frac{1}{a}\bar{e}_1^* + h_1\bar{f}_1^*),\ \bar{f}_2' = \frac{a}{(a+1)h_2h_3}(\frac{1}{a}\bar{e}_2^* + h_2\bar{f}_2^*),\ \bar{f}_3' = \frac{a}{(a+1)h_3^2}(\frac{1}{a}\bar{e}_3^* + h_3\bar{f}_3^*).$$

So, $\chi_1(h_1,h_2,h_3)$ is $Sp(V',f')$-equivalent with $\gamma_3(\frac{h_1}{h_3}, \frac{h_2}{h_3})$. ∎

5 Treatment of Case (B) of Corollary 3.18

In this section, we determine the $Sp(V,f)$-equivalence classes of trivectors that are contained in the subfamily of $\mathcal{E}_{\mathbb{F}'}$ corresponding to Case (B) of Corollary 3.18.

Suppose χ is a trivector of Type (E) of V which is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ such that $\chi = \phi\Big((\bar{u}_1+\delta\bar{u}_2)\wedge(\bar{u}_3+\delta\bar{u}_4)\wedge(\bar{u}_5+\delta\bar{u}_6)\Big)$ for some basis $\{\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5,\bar{u}_6\}$ of V satisfying $M_{f'}(\bar{u}_1+\delta\bar{u}_2,\bar{u}_3+\delta\bar{u}_4,\bar{u}_5+\delta\bar{u}_6) = M^*$ and $g(\bar{u}_1+\delta\bar{u}_2,\bar{u}_1+\delta\bar{u}_2)\neq 0$.

Lemma 5.1 *Let $\{\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5,\bar{v}_6\}$ be a basis of V such that $M_{f'}(\bar{v}_1+\delta\bar{v}_2,\bar{v}_3+\delta\bar{v}_4,\bar{v}_5+\delta\bar{v}_6) = M^*$ and $g(\bar{v}_1+\delta\bar{v}_2,\bar{v}_1+\delta\bar{v}_2)\neq 0$. Then there exists a basis $\{\bar{w}_1,\bar{w}_2,\bar{w}_3\}$ of $W=\langle\bar{v}_1+\delta\bar{v}_2,\bar{v}_3+\delta\bar{v}_4,\bar{v}_5+\delta\bar{v}_6\rangle$ such that $\bar{w}_1\wedge\bar{w}_2\wedge\bar{w}_3 = (\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)$, $M_{f'}(\bar{w}_1,\bar{w}_2,\bar{w}_3) = M^*$ and $M_g(\bar{w}_1,\bar{w}_2,\bar{w}_3)$ is diagonal. Moreover, if $M_g(\bar{w}_1,\bar{w}_2,\bar{w}_3) = \mathrm{diag}(k(a+1),h_1(a+1),h_2(a+1))$, then $k\neq 0$ and $h_1h_2(a+1)^2\neq 0$.*

Proof. Put $\bar{w}_1 := \bar{v}_1+\delta\bar{v}_2$ and let U denote the set of all vectors $\bar{u}\in W$ for which $g(\bar{u},\bar{w}_1)=0$. By Lemma 3.9, U is a subspace of W. In fact, it is a 2-dimensional subspace of W not containing the vector $\bar{w}_1$.

We prove that there exists a vector $\bar{w}_2\in U$ for which $g(\bar{w}_2,\bar{w}_2)\neq 0$. Suppose to the contrary that $g(\bar{u},\bar{u})=0$, $\forall\bar{u}\in U$. Let $\bar{w}_2'$ be an arbitrary vector of U and let $\bar{w}_3'$ be a vector of U such that $f'(\bar{w}_2',\bar{w}_3')=1$. Then $g(\bar{w}_2',\bar{w}_2') = g(\delta\bar{w}_3',\delta\bar{w}_3')=0$. By Lemmas 3.9 and 3.13, $g(\bar{w}_2'+\delta\bar{w}_3',\bar{w}_2'+\delta\bar{w}_3') = g(\bar{w}_2',\bar{w}_2')+g(\bar{w}_2',\delta\bar{w}_3')+g(\delta\bar{w}_3',\bar{w}_2')+g(\delta\bar{w}_3',\delta\bar{w}_3') = 0+\delta\cdot g(\bar{w}_2',\bar{w}_3')+(a+1)\cdot f'(\bar{w}_2',\bar{w}_3')+\delta\cdot g(\bar{w}_3',\bar{w}_2')+0 = (a+1)+\delta\cdot\Big(g(\bar{w}_2',\bar{w}_3')+g(\bar{w}_3',\bar{w}_2')\Big) = (a+1)+\delta(a+1)\cdot\mathsf{Im}(f'(\bar{w}_2',\bar{w}_3')) = a+1\neq 0$. So, we have a contradiction, proving that there must exist a $\bar{w}_2\in U$ for which $g(\bar{w}_2,\bar{w}_2)\neq 0$.

The set U' of all vectors $\bar{u}\in U$ for which $g(\bar{u},\bar{w}_2)=0$ is a subspace of U by Lemma 3.9. Since $g(\bar{w}_2,\bar{w}_2)\neq 0$, it is a 1-dimensional subspace of U. So, there must exist a unique $\bar{w}_3\in U'$ such that $\bar{w}_1\wedge\bar{w}_2\wedge\bar{w}_3 = (\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)$. We have $f'(\bar{w}_1,\bar{w}_2)=f'(\bar{w}_1,\bar{w}_3)=0$, $f'(\bar{w}_2,\bar{w}_3)\cdot\bar{w}_1 = \pi(\bar{w}_1\wedge\bar{w}_2\wedge\bar{w}_3) = \pi\Big((\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)\Big) = \bar{v}_1+\delta\bar{v}_2 = \bar{w}_1$ and hence $f'(\bar{w}_2,\bar{w}_3)=1$. So, $M_{f'}(\bar{w}_1,\bar{w}_2,\bar{w}_3)=M^*$. By Corollary 3.14, the matrix $M_g(\bar{w}_1,\bar{w}_2,\bar{w}_3)$ is symmetric. Since $g(\bar{w}_2,\bar{w}_1)=g(\bar{w}_3,\bar{w}_1)=g(\bar{w}_3,\bar{w}_2)=0$, one gets that $M_g(\bar{w}_1,\bar{w}_2,\bar{w}_3)$ is a diagonal matrix. If $M_g(\bar{w}_1,\bar{w}_2,\bar{w}_3) = \mathrm{diag}(k(a+1),h_1(a+1),h_2(a+1))$ for some $k,h_1,h_2\in\mathbb{F}$, then $M_{f'}(\bar{w}_1,\bar{w}_2,\bar{w}_3,\bar{w}_1^\tau,\bar{w}_2^\tau,\bar{w}_3^\tau)$ is equal to

$$\begin{bmatrix} 0 & 0 & 0 & k(a+1) & 0 & 0 \\ 0 & 0 & 1 & 0 & h_1(a+1) & 0 \\ 0 & 1 & 0 & 0 & 0 & h_2(a+1) \\ k(a+1) & 0 & 0 & 0 & 0 & 0 \\ 0 & h_1(a+1) & 0 & 0 & 0 & 1 \\ 0 & 0 & h_2(a+1) & 0 & 1 & 0 \end{bmatrix}$$

by Corollary 3.14. Since $M_{f'}(\bar{w}_1,\bar{w}_2,\bar{w}_3,\bar{w}_1^\tau,\bar{w}_2^\tau,\bar{w}_3^\tau)$ is nonsingular, we have $k\neq 0$ and $h_1h_2(a+1)^2\neq 1$. ∎

By Lemma 5.1, we may suppose that $\chi = \phi\Big((\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)\Big)$ where $\{\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5,\bar{v}_6\}$ is a basis of V such that $M_{f'}(\bar{v}_1+\delta\bar{v}_2,\bar{v}_3+\delta\bar{v}_4,\bar{v}_5+\delta\bar{v}_6)=M^*$ and $M_g(\bar{v}_1+\delta\bar{v}_2,\bar{v}_3+\delta\bar{v}_4,\bar{v}_5+\delta\bar{v}_6)=\mathrm{diag}(k(a+1),h_1(a+1),h_2(a+1))$ for some $k,h_1,h_2\in\mathbb{F}$ satisfying $k\neq 0$ and $h_1h_2(a+1)^2\neq 1$.

Since $k(a+1)=g(\bar{v}_1+\delta\bar{v}_2,\bar{v}_1+\delta\bar{v}_2)=f'(\bar{v}_1+\delta\bar{v}_2,\bar{v}_2+\delta\bar{v}_1)=(a+1)\cdot f'(\bar{v}_1,\bar{v}_2)$, we have $f'(\bar{v}_1,\bar{v}_2)=k$. In a similar way, one proves that $f'(\bar{v}_3,\bar{v}_4)=h_1$ and $f'(\bar{v}_5,\bar{v}_6)=h_2$.

Since $f'(\bar{v}_1+\delta\bar{v}_2,\bar{v}_3+\delta\bar{v}_4)=0$ and $g(\bar{v}_1+\delta\bar{v}_2,\bar{v}_3+\delta\bar{v}_4)=f'(\bar{v}_1+\delta\bar{v}_2,\bar{v}_4+\delta\bar{v}_3)=0$, we have $f'(\bar{v}_1+\delta\bar{v}_2,\bar{v}_3)=0$ and $f'(\bar{v}_1+\delta\bar{v}_2,\bar{v}_4)=0$ and hence

$$f'(\bar{v}_1,\bar{v}_3)=f'(\bar{v}_2,\bar{v}_3)=f'(\bar{v}_1,\bar{v}_4)=f'(\bar{v}_2,\bar{v}_4)=0.$$

In a similar way, one proves that

$$f'(\bar{v}_1,\bar{v}_5)=f'(\bar{v}_1,\bar{v}_6)=f'(\bar{v}_2,\bar{v}_5)=f'(\bar{v}_2,\bar{v}_6)=0.$$

Since $f'(\bar{v}_3+\delta\bar{v}_4,\bar{v}_5+\delta\bar{v}_6)=1$ and $g(\bar{v}_3+\delta\bar{v}_4,\bar{v}_5+\delta\bar{v}_6)=f'(\bar{v}_3+\delta\bar{v}_4,\bar{v}_6+\delta\bar{v}_5)=0$, we have $f'(\bar{v}_3+\delta\bar{v}_4,\bar{v}_5)=\frac{1}{a+1}$ and $f'(\bar{v}_3+\delta\bar{v}_4,\bar{v}_6)=\frac{\delta}{a+1}$ and hence

$$f'(\bar{v}_3,\bar{v}_5)=\frac{1}{a+1},\quad f'(\bar{v}_4,\bar{v}_5)=0,\quad f'(\bar{v}_3,\bar{v}_6)=0,\quad f'(\bar{v}_4,\bar{v}_6)=\frac{1}{a+1}.$$

So, $M_{f'}(\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5,\bar{v}_6)$ is equal to

$$\begin{bmatrix} 0 & k & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_1 & \frac{1}{a+1} & 0 \\ 0 & 0 & h_1 & 0 & 0 & \frac{1}{a+1} \\ 0 & 0 & \frac{1}{a+1} & 0 & 0 & h_2 \\ 0 & 0 & 0 & \frac{1}{a+1} & h_2 & 0 \end{bmatrix}.$$

Now, put $\bar{v}'_1:=\bar{v}_1$, $\bar{v}'_2:=\bar{v}_2$, $\bar{v}'_3:=\bar{v}_3+h_1(a+1)\bar{v}_6$, $\bar{v}'_4:=\bar{v}_4$, $\bar{v}'_5:=\bar{v}_5$ and $\bar{v}'_6:=h_2(a+1)\bar{v}_3+\bar{v}_6$. Then $M_{f'}(\bar{v}'_1,\bar{v}'_2,\bar{v}'_3,\bar{v}'_4,\bar{v}'_5,\bar{v}'_6)$ is equal to

$$\begin{bmatrix} 0 & k & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{h_1h_2(a+1)^2+1}{a+1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{h_1h_2(a+1)^2+1}{a+1} \\ 0 & 0 & \frac{h_1h_2(a+1)^2+1}{a+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h_1h_2(a+1)^2+1}{a+1} & 0 & 0 \end{bmatrix}.$$

So, there exists a hyperbolic basis $(\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\bar{e}_3,\bar{f}_3)$ of (V,f) such that

$$\bar{v}'_1=\bar{e}_1,\bar{v}'_2=k\bar{f}_1,\bar{v}'_3=\frac{h_1h_2(a+1)^2+1}{a+1}\bar{e}_2,\bar{v}'_4=\bar{e}_3,\bar{v}'_5=\bar{f}_2,\bar{v}'_6=\frac{h_1h_2(a+1)^2+1}{a+1}\bar{f}_3.$$

Then

$$\bar{v}_1 = \bar{e}_1, \bar{v}_2 = k\bar{f}_1, \bar{v}_3 = \frac{1}{a+1}(\bar{e}_2 + h_1(a+1)\bar{f}_3), \bar{v}_4 = \bar{e}_3, \bar{v}_5 = \bar{f}_2, \bar{v}_6 = \frac{1}{a+1}(h_2(a+1)\bar{e}_2 + \bar{f}_3).$$

So,

$$\begin{aligned}
\chi &= \frac{a+1}{a} \cdot \bar{v}_1 \wedge \bar{v}_3 \wedge \bar{v}_5 + (a+1) \cdot \bar{v}_2 \wedge \bar{v}_4 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_2) \wedge (\bar{v}_3 + \bar{v}_4) \wedge (\bar{v}_5 + \bar{v}_6) \\
&= \frac{1}{a} \cdot \bar{e}_1 \wedge (\bar{e}_2 + h_1(a+1)\bar{f}_3) \wedge \bar{f}_2 + k \cdot \bar{f}_1 \wedge \bar{e}_3 \wedge (h_2(a+1)\bar{e}_2 + \bar{f}_3) + \frac{1}{(a+1)^2} \cdot \\
&\quad (\bar{e}_1 + k\bar{f}_1) \wedge (\bar{e}_2 + (a+1)\bar{e}_3 + h_1(a+1)\bar{f}_3) \wedge (h_2(a+1)\bar{e}_2 + (a+1)\bar{f}_2 + \bar{f}_3).
\end{aligned}$$

Hence, χ is $Sp(V,f)$-equivalent with the trivector $\chi_2(k, h_1, h_2)$.

Reversing the above procedure, we see that the trivector $\chi_2(k, h_1, h_2)$ is of the form $\phi\Big((\bar{v}_1^* + \delta\bar{v}_2^*) \wedge (\bar{v}_3^* + \delta\bar{v}_4^*) \wedge (\bar{v}_5^* + \delta\bar{v}_6^*)\Big)$ where $\{\bar{v}_1^*, \bar{v}_2^*, \bar{v}_3^*, \bar{v}_4^*, \bar{v}_5^*, \bar{v}_6^*\}$ is some basis of V satisfying $M_{f'}(\bar{v}_1^* + \delta\bar{v}_2^*, \bar{v}_3^* + \delta\bar{v}_4^*, \bar{v}_5^* + \delta\bar{v}_6^*) = M^*$ and $M_g(\bar{v}_1^* + \delta\bar{v}_2^*, \bar{v}_3^* + \delta\bar{v}_4^*, \bar{v}_5^* + \delta\bar{v}_6^*) = \mathrm{diag}(k(a+1), h_1(a+1), h_2(a+1))$. So, $\chi_2(k, h_1, h_2)$ is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ by Lemma 3.4.

We will now determine under which conditions two trivectors of Type (E2') are $Sp(V,f)$-equivalent. Let $k, h_1, h_2, k', h_1', h_2' \in \mathbb{F}$ with $k \neq 0 \neq k'$ and $h_1h_2(a+1)^2 \neq 1 \neq h_1'h_2'(a+1)^2$. By Lemma 3.12, the two trivectors $\chi_2(k, h_1, h_2)$ and $\chi_2(k', h_1', h_2')$ are $Sp(V,f)$-equivalent if and only if there exists a matrix $A \in SL(3, \mathbb{F}')$ such that

$$M^* = A \cdot M^* \cdot A^{\mathrm{T}},$$

$$\begin{aligned}
\mathrm{diag}(k'(a+1), h_1'(a+1), h_2'(a+1)) = A \cdot \mathrm{diag}(k(a+1), h_1(a+1), h_2(a+1)) \cdot A^{\mathrm{T}} \\
+(a+1) \cdot A \cdot M^* \cdot \mathsf{Im}(A)^{\mathrm{T}}.
\end{aligned}$$

Now, the latter condition is equivalent to

$$\mathrm{diag}(k', h_1', h_2') = A \cdot \mathrm{diag}(k, h_1, h_2) \cdot A^{\mathrm{T}} + A \cdot M^* \cdot \mathsf{Im}(A)^{\mathrm{T}}.$$

By Lemma 3.15, A has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where $a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} \in \mathbb{F}$ such that $a_{22}a_{33} - a_{23}a_{32} = 1$. One now computes that $A \cdot \mathrm{diag}(k, h_1, h_2) \cdot A^{\mathrm{T}}$ is equal to

$$\begin{bmatrix} k & ka_{21} & ka_{31} \\ ka_{21} & ka_{21}^2 + h_1a_{22}^2 + h_2a_{23}^2 & ka_{21}a_{31} + h_1a_{22}a_{32} + h_2a_{23}a_{33} \\ ka_{31} & ka_{31}a_{21} + h_1a_{32}a_{22} + h_2a_{33}a_{23} & ka_{31}^2 + h_1a_{32}^2 + h_2a_{33}^2 \end{bmatrix}$$

and that

$$A \cdot M^* \cdot \mathsf{Im}(A)^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{23} \cdot \mathsf{Im}(a_{22}) + a_{22} \cdot \mathsf{Im}(a_{23}) & a_{23} \cdot \mathsf{Im}(a_{32}) + a_{22} \cdot \mathsf{Im}(a_{33}) \\ 0 & a_{33} \cdot \mathsf{Im}(a_{22}) + a_{32} \cdot \mathsf{Im}(a_{23}) & a_{33} \cdot \mathsf{Im}(a_{32}) + a_{32} \cdot \mathsf{Im}(a_{33}) \end{bmatrix}.$$

The condition $\mathrm{diag}(k', h_1', h_2') = A \cdot \mathrm{diag}(k, h_1, h_2) \cdot A^{\mathrm{T}} + A \cdot M^* \cdot \mathsf{Im}(A)^{\mathrm{T}}$ is then equivalent with $k' = k$, $a_{21} = a_{31} = 0$ and

$$\begin{bmatrix} h_1' & 0 \\ 0 & h_2' \end{bmatrix} = B \cdot \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \cdot B^{\mathrm{T}} + B \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathsf{Im}(B)^{\mathrm{T}},$$

where B is the matrix $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$. So, we have

Proposition 5.2 *Let $k, h_1, h_2, k', h_1', h_2' \in \mathbb{F}$ with $k \neq 0 \neq k'$ and $h_1 h_2 (a+1)^2 \neq 1 \neq h_1' h_2' (a+1)^2$. Then the trivectors $\chi_2(k, h_1, h_2)$ and $\chi_2(k', h_1', h_2')$ are $Sp(V, f)$-equivalent if and only if $k = k'$ and there exists a matrix $A \in SL(2, \mathbb{F}')$ over $\mathbb{F}'$ such that*

$$\begin{bmatrix} h_1' & 0 \\ 0 & h_2' \end{bmatrix} = A \cdot \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \cdot A^{\mathrm{T}} + A \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathsf{Im}(A)^{\mathrm{T}}.$$

Lemma 5.3 *Let $h_1, h_2, h_1', h_2' \in \mathbb{F}$ and $\alpha, \beta, \gamma, \nu \in \mathbb{F}'$. Put $A := \begin{bmatrix} \alpha & \beta \\ \gamma & \nu \end{bmatrix}$. Then the conditions*

$$\begin{cases} \det(A) & = & 1, \\ \begin{bmatrix} h_1' & 0 \\ 0 & h_2' \end{bmatrix} & = & A \cdot \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \cdot A^{\mathrm{T}} + A \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathsf{Im}(A)^{\mathrm{T}} \end{cases}$$

are equivalent with each of the following two equivalent sets of equations:

$$\begin{cases} h_1' = \alpha^2 h_1 + \beta^2 h_2 + \mathsf{Im}(\alpha\beta), \\ h_2' = \gamma^2 h_1 + \nu^2 h_2 + \mathsf{Im}(\gamma\nu), \\ (\beta h_2 + \mathsf{Im}(\alpha)) \cdot \nu + (\alpha h_1 + \mathsf{Im}(\beta)) \cdot \gamma = 0, \\ \alpha \cdot \nu + \beta \cdot \gamma = 1, \end{cases}$$

$$\begin{cases} h_1' = \alpha^2 h_1 + \beta^2 h_2 + \mathsf{Im}(\alpha\beta), \\ h_2' = \gamma^2 h_1 + \nu^2 h_2 + \mathsf{Im}(\gamma\nu), \\ (\gamma h_1 + \mathsf{Im}(\nu)) \cdot \alpha + (\nu h_2 + \mathsf{Im}(\gamma)) \cdot \beta = 0, \\ \nu \cdot \alpha + \gamma \cdot \beta = 1. \end{cases}$$

Proof. One can easily verify that $\mathsf{Im}(\lambda_1 \lambda_2) = \lambda_1 \cdot \mathsf{Im}(\lambda_2) + \lambda_2 \cdot \mathsf{Im}(\lambda_1)$ for all $\lambda_1, \lambda_2 \in \mathbb{F}'$. Taking this fact into account, a straightforward computation gives that the conditions of the lemma are equivalent with:

(1) $h_1' = \alpha^2 h_1 + \beta^2 h_2 + \mathsf{Im}(\alpha\beta)$,

(2) $h_2' = \gamma^2 h_1 + \nu^2 h_2 + \mathsf{Im}(\gamma\nu)$,
(3) $\alpha\nu + \beta\gamma = 1$,
(4) $(\beta h_2 + \mathsf{Im}(\alpha)) \cdot \nu + (\alpha h_1 + \mathsf{Im}(\beta)) \cdot \gamma = 0$,
(5) $(\gamma h_1 + \mathsf{Im}(\nu)) \cdot \alpha + (\nu h_2 + \mathsf{Im}(\gamma)) \cdot \beta = 0$.

Now, $(\beta h_2 + \mathsf{Im}(\alpha)) \cdot \nu + (\alpha h_1 + \mathsf{Im}(\beta)) \cdot \gamma + (\gamma h_1 + \mathsf{Im}(\nu)) \cdot \alpha + (\nu h_2 + \mathsf{Im}(\gamma)) \cdot \beta = \mathsf{Im}(\alpha)\cdot\nu + \mathsf{Im}(\nu)\cdot\alpha + \mathsf{Im}(\beta)\cdot\gamma + \mathsf{Im}(\gamma)\cdot\beta = \mathsf{Im}(\alpha\nu) + \mathsf{Im}(\beta\gamma) = \mathsf{Im}(\alpha\nu+\beta\gamma)$. So, assuming the validity of (3), we see that (4) and (5) are equivalent. ■

Lemma 5.4 *Let $h_1, h_2, h_1', h_2' \in \mathbb{F}$ and $\alpha, \beta, \gamma, \nu \in \mathbb{F}'$ such that $\alpha\nu + \beta\gamma = 1$ and*

$$\begin{bmatrix} h_1' & 0 \\ 0 & h_2' \end{bmatrix} = A \cdot \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \cdot A^{\mathrm{T}} + A \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \mathsf{Im}(A)^{\mathrm{T}},$$

where $A := \begin{bmatrix} \alpha & \beta \\ \gamma & \nu \end{bmatrix}$. Then $h_1 h_2 = h_1' h_2'$.

Proof. We make use of the first set of equations given in Lemma 5.3. Consider the following linear system of two variables γ and ν:

$$\begin{cases} (\beta h_2 + \mathsf{Im}(\alpha)) \cdot \nu + (\alpha h_1 + \mathsf{Im}(\beta)) \cdot \gamma & = & 0, \\ \alpha \cdot \nu + \beta \cdot \gamma & = & 1. \end{cases}$$

The determinant of this system is equal to h_1' and we have that

$$\nu h_1' = \alpha h_1 + \mathsf{Im}(\beta), \qquad \gamma h_1' = \beta h_2 + \mathsf{Im}(\alpha).$$

Using this, we find

$$\begin{aligned} (h_1')^2 h_2' &= (\gamma h_1')^2 h_1 + (\nu h_1')^2 h_2 + \mathsf{Im}((\gamma h_1')(\nu h_1')) \\ &= (\beta h_2 + \mathsf{Im}(\alpha))^2 h_1 + (\alpha h_1 + \mathsf{Im}(\beta))^2 h_2 + \mathsf{Im}((\beta h_2 + \mathsf{Im}(\alpha))(\alpha h_1 + \mathsf{Im}(\beta))) \\ &= \beta^2 h_2^2 h_1 + \mathsf{Im}(\alpha)^2 h_1 + \alpha^2 h_1^2 h_2 + \mathsf{Im}(\beta)^2 h_2 + h_1 h_2 \cdot \mathsf{Im}(\alpha\beta) + \mathsf{Im}(\alpha)^2 h_1 \\ &\qquad + \mathsf{Im}(\beta)^2 h_2 \\ &= h_1 h_2 (\alpha^2 h_1 + \beta^2 h_2 + \mathsf{Im}(\alpha\beta)) \\ &= h_1 h_2 h_1'. \end{aligned}$$

So, if $h_1' \neq 0$, then $h_1 h_2 = h_1' h_2'$. If $h_1' = 0$ and one of h_1, h_2 is 0, then also $h_1 h_2 = h_1' h_2'$.

Suppose therefore that $h_1' = 0$, $h_1 \neq 0$ and $h_2 \neq 0$. Then $\beta h_2 + \mathsf{Im}(\alpha) = 0$ and $\alpha h_1 + \mathsf{Im}(\beta) = 0$. So, $\alpha = \frac{\mathsf{Im}(\beta)}{h_1} \in \mathbb{F}$ and $\beta = \frac{\mathsf{Im}(\alpha)}{h_2} \in \mathbb{F}$. It follows that $\mathsf{Im}(\alpha) = \mathsf{Im}(\beta) = 0$ and hence that $\alpha = \beta = 0$. This is however impossible since $\alpha\nu + \beta\gamma = 1$. ■

Lemma 5.5 (1) *Let $k, h_1, h_2 \in \mathbb{F}$ such that $k \neq 0$ and $h_1 h_2 (a+1)^2 \neq 1$. Then the trivector $\chi_2(k, h_1, h_2)$ is $Sp(V, f)$-equivalent with the trivector $\chi_2(k, h_2, h_1)$.*

(2) *Let $k, h_1 \in \mathbb{F}$ with $k \neq 0$. Then the trivector $\chi_2(k, 0, 0)$ is $Sp(V, f)$-equivalent with both $\chi_2(k, h_1, 0)$ and $\chi_2(k, 0, h_1)$.*

Proof. (1) This follows from symmetry, or alternatively, one can take $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in Proposition 5.2.

(2) We may suppose that $h_1 \neq 0$. By part (1), it suffices to prove that $\chi_2(k,0,0)$ and $\chi_2(k,h_1,0)$ are $Sp(V,f)$-equivalent. By Lemma 5.3, it suffices to prove that there exist $\alpha, \beta, \gamma, \nu$ satisfying

$$h_1 = \mathsf{Im}(\alpha\beta), \quad 0 = \mathsf{Im}(\gamma\nu), \quad 0 = \mathsf{Im}(\alpha) \cdot \nu + \mathsf{Im}(\beta) \cdot \gamma, \quad 1 = \alpha\nu + \beta\gamma.$$

Now, take α and β arbitrary such that $h_1 = \mathsf{Im}(\alpha\beta)$ and put $\nu = \frac{\mathsf{Im}(\beta)}{\mathsf{Im}(\alpha\beta)} = \frac{\mathsf{Im}(\beta)}{h_1}$ and $\gamma = \frac{\mathsf{Im}(\alpha)}{\mathsf{Im}(\alpha\beta)} = \frac{\mathsf{Im}(\alpha)}{h_1}$. Then all required conditions are satisfied. ∎

Proposition 5.6 *Let $k, h_1, h_2, k', h_1', h_2' \in \mathbb{F}$ with $k \neq 0 \neq k'$ and $h_1 h_2 (a+1)^2 \neq 1 \neq h_1' h_2' (a+1)^2$. Then $\chi_2(k,h_1,h_2)$ and $\chi_2(k',h_1',h_2')$ are $Sp(V,f)$-equivalent if and only if $k = k'$, $h_1 h_2 = h_1' h_2'$ and there exist $\alpha, \beta \in \mathbb{F}'$ such that $h_1' = \alpha^2 h_1 + \beta^2 h_2 + \mathsf{Im}(\alpha\beta)$.*

Proof. By Proposition 5.2, Lemma 5.3 and Lemma 5.4, these conditions are necessary.

Conversely, suppose that $k = k'$, $h_1 h_2 = h_1' h_2'$ and there exist $\alpha, \beta \in \mathbb{F}'$ such that $h_1' = \alpha^2 h_1 + \beta^2 h_2 + \mathsf{Im}(\alpha\beta)$. We need to prove that $\chi_2(k,h_1,h_2)$ and $\chi_2(k',h_1',h_2')$ are $Sp(V,f)$-equivalent. By Lemma 5.5(2) and the fact that $h_1 h_2 = h_1' h_2'$, we may suppose that h_1, h_2, h_1', h_2' are distinct from 0. The linear system

$$\begin{cases} (\beta h_2 + \mathsf{Im}(\alpha)) \cdot \nu + (\alpha h_1 + \mathsf{Im}(\beta)) \cdot \gamma & = & 0, \\ \alpha \cdot \nu + \beta \cdot \gamma & = & 1 \end{cases}$$

has a unique solution for γ and ν, since the determinant of the system is equal to $\alpha^2 h_1 + \beta^2 h_2 + \mathsf{Im}(\alpha\beta) = h_1' \neq 0$. If we put $h_2'' := \gamma^2 h_1 + \nu^2 h_2 + \mathsf{Im}(\gamma\nu)$, then by Proposition 5.2 and Lemma 5.3, $\chi_2(k,h_1,h_2)$ and $\chi_2(k',h_1',h_2'')$ are $Sp(V,f)$-equivalent. This implies that $h_1 h_2 = h_1' h_2''$ by Lemma 5.4. Since also $h_1 h_2 = h_1' h_2'$, we have $h_2' = h_2''$. Hence, $\chi_2(k,h_1,h_2)$ and $\chi_2(k',h_1',h_2')$ are $Sp(V,f)$-equivalent. ∎

The following corollary to Proposition 5.6 is precisely Theorem 1.3(3).

Corollary 5.7 *Let $k, h_1, h_2, k', h_1', h_2' \in \mathbb{F}$ with $k \neq 0 \neq k'$ and $h_1 h_2 (a+1)^2 \neq 1 \neq h_1' h_2' (a+1)^2$. Then $\chi_2(k,h_1,h_2)$ and $\chi_2(k',h_1',h_2')$ are $Sp(V,f)$-equivalent if and only if $k = k'$, $h_1 h_2 = h_1' h_2'$ and there exist $X, Y, Z, U \in \mathbb{F}$ such that $h_1' = h_1(X^2 + aY^2) + h_2(Z^2 + aU^2) + (XU + YZ)$.*

Proof. Put $\alpha = X + \delta Y$ and $\beta = Z + \delta U$ in Proposition 5.6. ∎

As before, let $k \in \mathbb{F}^*$ and $h_1, h_2 \in \mathbb{F}$ such that $h_1 h_2 (a+1)^2 \neq 1$. By Corollary 3.6, we know that the trivector $\chi_2(k,h_1,h_2)$ is a trivector of Type (D) when regarded as a trivector of V'. One can now ask to which of the trivectors mentioned in Section 2 $\chi_2(k,h_1,h_2)$ is $Sp(V',f')$-equivalent to. The following proposition answers this question.

Proposition 5.8 *Let $k \in \mathbb{F}^*$ and $h_1, h_2 \in \mathbb{F}$ such that $h_1h_2(a+1)^2 \neq 1$. Then the trivector $\chi_2(k, h_1, h_2)$ is $Sp(V', f')$-equivalent with the trivector $\gamma_2(\lambda)$ of V', where $\lambda = \frac{k^2}{a^2}(1 + h_1h_2(a+1)^2)$.*

Proof. Put $\chi := \chi_2(k, h_1, h_2)$. Then

$$\begin{aligned}
\chi &= \Big(\bar{e}_1^* + \delta k \bar{f}_1^*\Big) \wedge \Big(\frac{\bar{e}_2^*}{a+1} + h_1 \bar{f}_3^* + \delta \bar{e}_3^*\Big) \wedge \Big(\frac{\bar{f}_2^*}{a} + h_2 \bar{e}_2^* + \frac{\bar{f}_3^*}{a+1}\Big) \\
&+ \Big(\frac{\bar{e}_2^*}{a+1} + h_1 \bar{f}_3^* + \delta \bar{e}_3^*\Big) \wedge \Big(\bar{f}_2^* + h_2 \delta \bar{e}_2^* + \frac{\delta}{a+1} \bar{f}_3^*\Big) \wedge \Big(\frac{\bar{e}_1^*}{a} + k \bar{f}_1^*\Big) \\
&+ \Big(\bar{f}_2^* + h_2 \delta \bar{e}_2^* + \frac{\delta}{a+1} \bar{f}_3^*\Big) \wedge \Big(\bar{e}_1^* + \delta k \bar{f}_1^*\Big) \wedge \Big(\frac{\bar{e}_2^*}{a(a+1)} + \frac{h_1}{a} \bar{f}_3^* + \bar{e}_3^*\Big).
\end{aligned}$$

Since $f'(\bar{e}_1^* + \delta k \bar{f}_1^*, \frac{\bar{e}_2^*}{a+1} + h_1 \bar{f}_3^* + \delta \bar{e}_3^*) = 0$, $f'(\bar{e}_1^* + \delta k \bar{f}_1^*, \bar{f}_2^* + h_2 \delta \bar{e}_2^* + \frac{\delta}{a+1} \bar{f}_3^*) = 0$ and $f'(\frac{\bar{e}_2^*}{a+1} + h_1 \bar{f}_3^* + \delta \bar{e}_3^*, \bar{f}_2^* + h_2 \delta \bar{e}_2^* + \frac{\delta}{a+1} \bar{f}_3^*) = 1$, the base 3-space of the trivector χ of V' is not totally isotropic. So, χ is $Sp(V', f')$-equivalent with either γ_1 or $\gamma_2(\lambda)$ for some $\lambda \in \mathbb{F}^*$. We also have

$$\begin{aligned}
\chi &= \frac{1}{a} \cdot \bar{e}_1^* \wedge (\bar{e}_2^* + h_1(a+1)\bar{f}_3^*) \wedge \bar{f}_2^* + k \cdot \bar{f}_1^* \wedge \bar{e}_3^* \wedge (h_2(a+1)\bar{e}_2^* + \bar{f}_3^*) + \frac{1}{(a+1)^2} \cdot \\
&\quad (\bar{e}_1^* + k\bar{f}_1^*) \wedge (\bar{e}_2^* + (a+1)\bar{e}_3^* + h_1(a+1)\bar{f}_3^*) \wedge (h_2(a+1)\bar{e}_2^* + (a+1)\bar{f}_2^* + \bar{f}_3^*).
\end{aligned}$$

We now compute $\pi(\chi \wedge \pi(\chi))$. We have

$$\pi(\chi) = \frac{1}{a}\bar{e}_1^* + k\bar{f}_1^*$$

and

$$\begin{aligned}
\chi \wedge \pi(\chi) &= \frac{k}{a} \cdot \bar{e}_1^* \wedge \bar{f}_1^* \wedge \bar{e}_3^* \wedge \Big(h_2(a+1)\bar{e}_2^* + \bar{f}_3^*\Big) + \frac{k}{a} \cdot \bar{e}_1^* \wedge \bar{f}_1^* \wedge \Big(\bar{e}_2^* + h_1(a+1)\bar{f}_3^*\Big) \wedge \\
&\quad \bar{f}_2^* + \frac{k}{a(a+1)} \cdot \bar{e}_1^* \wedge \bar{f}_1^* \wedge \Big(\bar{e}_2^* + (a+1)\bar{e}_3^* + h_1(a+1)\bar{f}_3^*\Big) \wedge \\
&\quad \Big(h_2(a+1)\bar{e}_2^* + (a+1)\bar{f}_2^* + \bar{f}_3^*\Big) \\
&= \bar{e}_1^* \wedge \bar{f}_1^* \wedge \Big(\frac{k(1 + h_1h_2(a+1)^2)}{a(a+1)} \cdot \bar{e}_2^* \wedge \bar{f}_3^* + \frac{k(a+1)}{a} \cdot \bar{e}_3^* \wedge \bar{f}_2^*\Big).
\end{aligned}$$

So,

$$\pi(\chi \wedge \pi(\chi)) = \frac{k(1 + h_1h_2(a+1)^2)}{a(a+1)} \cdot \bar{e}_2^* \wedge \bar{f}_3^* + \frac{k(a+1)}{a} \cdot \bar{e}_3^* \wedge \bar{f}_2^*.$$

Since $\pi(\chi) \wedge \pi(\chi \wedge \pi(\chi))$ is not a completely decomposable trivector, χ cannot be $Sp(V', f')$-equivalent with γ_1 by Section 4 of [6]. So, χ is $Sp(V', f')$-equivalent with $\gamma_2(\lambda)$ for some $\lambda \in \mathbb{F}^*$. By Section 4 of [6], the precise value of λ is obtained by multiplying the coefficients of $\bar{e}_2^* \wedge \bar{f}_3^*$ and $\bar{e}_3^* \wedge \bar{f}_2^*$ in $\pi(\chi \wedge \pi(\chi))$. So, $\lambda = \frac{k^2(1+h_1h_2(a+1)^2)}{a^2}$. ■

6 Treatment of Case (C) of Corollary 3.18

In this section, we determine the $Sp(V,f)$-equivalence classes of trivectors that are contained in the subfamily of $\mathcal{E}_{\mathbb{F}'}$ corresponding to Case (C) of Corollary 3.18.

Suppose χ is a trivector of Type (E) of V which is $GL(V)$-equivalent with $\chi^*_{\mathbb{F}'}$ such that $\chi = \phi\Big((\bar{u}_1+\delta\bar{u}_2)\wedge(\bar{u}_3+\delta\bar{u}_4)\wedge(\bar{u}_5+\delta\bar{u}_6)\Big)$ for some basis $\{\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4,\bar{u}_5,\bar{u}_6\}$ of V satisfying $M_{f'}(\bar{u}_1+\delta\bar{u}_2,\bar{u}_3+\delta\bar{u}_4,\bar{u}_5+\delta\bar{u}_6)=M^*$ and $g(\bar{u}_1+\delta\bar{u}_2,\bar{u}_1+\delta\bar{u}_2)=0$.

Lemma 6.1 *Let $\{\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4,\bar{v}_5,\bar{v}_6\}$ be a basis of V such that $M_{f'}(\bar{v}_1+\delta\bar{v}_2,\bar{v}_3+\delta\bar{v}_4,\bar{v}_5+\delta\bar{v}_6)=M^*$ and $g(\bar{v}_1+\delta\bar{v}_2,\bar{v}_1+\delta\bar{v}_2)=0$. Let k be an arbitrary element of $\mathbb{F}'\setminus\{0\}$. Then there exists a basis $\{\bar{w}_1,\bar{w}_2,\bar{w}_3\}$ of $W:=\langle\bar{v}_1+\delta\bar{v}_2,\bar{v}_3+\delta\bar{v}_4,\bar{v}_5+\delta\bar{v}_6\rangle$ such that $\bar{w}_1\wedge\bar{w}_2\wedge\bar{w}_3=(\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)$, $M_{f'}(\bar{w}_1,\bar{w}_2,\bar{w}_3)=M^*$ and $M_g(\bar{w}_1,\bar{w}_2,\bar{w}_3)$ is of the form*

$$\begin{bmatrix} 0 & 0 & k \\ 0 & \lambda_1 & 0 \\ k & 0 & \lambda_2 \end{bmatrix}$$

for some $\lambda_1,\lambda_2\in\mathbb{F}$ with $\lambda_1\neq 0$.

Proof. Put $\bar{w}_1:=\bar{v}_1+\delta\bar{v}_2$. Let U denote the set of all vectors $\bar{u}\in W$ for which $g(\bar{u},\bar{w}_1)=0$. Then U is a subspace of W by Lemma 3.9. By Lemma 3.13(2), U also consists of all vectors $\bar{u}\in W$ for which $g(\bar{w}_1,\bar{u})=0$. If $U=W$, then we would have $f'(\bar{w}_1,\bar{v})=0$ for all $\bar{v}\in V$, clearly a contradiction. So, U is a 2-dimensional subspace of W. By Lemma 3.9(3), there exists a vector $\bar{w}_3\in W\setminus U$ such that $g(\bar{w}_3,\bar{w}_1)=k$. Then also $g(\bar{w}_1,\bar{w}_3)=k$ by Lemma 3.13(2). Now, let U' denote the set of all vectors $\bar{u}\in W$ for which $g(\bar{u},\bar{w}_3)=0$. Then U' is a subspace of W (by Lemma 3.9) which should be two-dimensional since $\bar{w}_1\notin U'$. Since $\bar{w}_1\in U$ and $\bar{w}_1\notin U'$, $U\cap U'$ is a one-dimensional subspace. Let $\bar{w}_2$ be the unique vector of $U\cap U'$ such that $(\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)=\bar{w}_1\wedge\bar{w}_2\wedge\bar{w}_3$. Since $f'(\bar{w}_2,\bar{w}_3)\cdot\bar{w}_1=\pi(\bar{w}_1\wedge\bar{w}_2\wedge\bar{w}_3)=\pi\Big((\bar{v}_1+\delta\bar{v}_2)\wedge(\bar{v}_3+\delta\bar{v}_4)\wedge(\bar{v}_5+\delta\bar{v}_6)\Big)=\bar{v}_1+\delta\bar{v}_2=\bar{w}_1$, we have $f'(\bar{w}_2,\bar{w}_3)=1$. So, $M_g(\bar{w}_1,\bar{w}_2,\bar{w}_3)$ should be a symmetric matrix by Corollary 3.14. Since $g(\bar{w}_1,\bar{w}_1)=g(\bar{w}_2,\bar{w}_1)=g(\bar{w}_2,\bar{w}_3)=0$ and $g(\bar{w}_3,\bar{w}_1)=k$, this implies that $M_g(\bar{w}_1,\bar{w}_2,\bar{w}_3)$ has the form

$$\begin{bmatrix} 0 & 0 & k \\ 0 & \lambda_1 & 0 \\ k & 0 & \lambda_2 \end{bmatrix}$$

for some $\lambda_1,\lambda_2\in\mathbb{F}$. By Corollary 3.14, we have that $M_{f'}(\bar{w}_1,\bar{w}_2,\bar{w}_3,\bar{w}_1^\tau,\bar{w}_2^\tau,\bar{w}_3^\tau)$ is equal to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & k \\ 0 & 0 & 1 & 0 & \lambda_1 & 0 \\ 0 & 1 & 0 & k & 0 & \lambda_2 \\ 0 & 0 & k & 0 & 0 & (a+1)\cdot\mathsf{Im}(k) \\ 0 & \lambda_1 & 0 & 0 & 0 & 1 \\ k & 0 & \lambda_2 & (a+1)\cdot\mathsf{Im}(k) & 1 & 0 \end{bmatrix}.$$

Since the determinant of $M_{f'}(\bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_1^\tau, \bar{w}_2^\tau, \bar{w}_3^\tau)$ is distinct from 0, we have that $\lambda_1 \neq 0$. ∎

Now, let k be a fixed element of $\mathbb{F}^*$, to be determined later. By Lemma 6.1, we may suppose that $\chi = \phi\Big((\bar{v}_1 + \delta\bar{v}_2) \wedge (\bar{v}_3 + \delta\bar{v}_4) \wedge (\bar{v}_5 + \delta\bar{v}_6)\Big)$ where $\{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6\}$ is a basis of V such that $M_{f'}(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = M^*$ and

$$M_g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = \begin{bmatrix} 0 & 0 & k(a+1) \\ 0 & h_1(a+1) & 0 \\ k(a+1) & 0 & h_2(a+1) \end{bmatrix},$$

for some $h_1, h_2 \in \mathbb{F}$ with $h_1 \neq 0$.

Since $g(\bar{v}_1+\delta\bar{v}_2, \bar{v}_1+\delta\bar{v}_2) = 0$, $g(\bar{v}_3+\delta\bar{v}_4, \bar{v}_3+\delta\bar{v}_4) = h_1(a+1)$ and $g(\bar{v}_5+\delta\bar{v}_6, \bar{v}_5+\delta\bar{v}_6) = h_2(a+1)$, we have

$$f'(\bar{v}_1, \bar{v}_2) = 0, \qquad f'(\bar{v}_3, \bar{v}_4) = h_1, \qquad f'(\bar{v}_5, \bar{v}_6) = h_2.$$

From $f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4) = 0$ and $f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_4 + \delta\bar{v}_3) = g(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_3 + \delta\bar{v}_4) = 0$, we find

$$f'(\bar{v}_1, \bar{v}_3) = f'(\bar{v}_1, \bar{v}_4) = f'(\bar{v}_2, \bar{v}_3) = f'(\bar{v}_2, \bar{v}_4) = 0.$$

Since $f'(\bar{v}_1+\delta\bar{v}_2, \bar{v}_5+\delta\bar{v}_6) = 0$ and $f'(\bar{v}_1+\delta\bar{v}_2, \bar{v}_6+\delta\bar{v}_5) = g(\bar{v}_1+\delta\bar{v}_2, \bar{v}_5+\delta\bar{v}_6) = k(a+1)$, we have $f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_6) = k$ and $f'(\bar{v}_1 + \delta\bar{v}_2, \bar{v}_5) = k\delta$ and hence

$$f'(\bar{v}_1, \bar{v}_5) = 0, \qquad f'(\bar{v}_1, \bar{v}_6) = k, \qquad f'(\bar{v}_2, \bar{v}_5) = k, \qquad f'(\bar{v}_2, \bar{v}_6) = 0.$$

Similarly as in the treatment of Case (B), the facts that $f'(\bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = 1$ and $f'(\bar{v}_3 + \delta\bar{v}_4, \bar{v}_6 + \delta\bar{v}_5) = g(\bar{v}_3 + \delta\bar{v}_4, \bar{v}_5 + \delta\bar{v}_6) = 0$ imply that

$$f'(\bar{v}_3, \bar{v}_5) = \frac{1}{a+1}, \qquad f'(\bar{v}_4, \bar{v}_5) = 0, \qquad f'(\bar{v}_3, \bar{v}_6) = 0, \qquad f'(\bar{v}_4, \bar{v}_6) = \frac{1}{a+1}.$$

So, $M_{f'}(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5, \bar{v}_6)$ is equal to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & h_1 & \frac{1}{a+1} & 0 \\ 0 & 0 & h_1 & 0 & 0 & \frac{1}{a+1} \\ 0 & k & \frac{1}{a+1} & 0 & 0 & h_2 \\ k & 0 & 0 & \frac{1}{a+1} & h_2 & 0 \end{bmatrix}.$$

Now, put

$$\bar{v}_1' := \bar{v}_1, \quad \bar{v}_2' := \bar{v}_2, \quad \bar{v}_3' := (a+1)k\bar{v}_3+\bar{v}_2, \quad \bar{v}_4' := (a+1)k\bar{v}_4+\bar{v}_1, \quad \bar{v}_5' := \bar{v}_5+\frac{h_2}{k}\bar{v}_1, \quad \bar{v}_6' := \bar{v}_6.$$

Easy formulas are obtained if we put $k := \frac{1}{a+1}$. Then $M_{f'}(\bar{v}_1', \bar{v}_2', \bar{v}_3', \bar{v}_4', \bar{v}_5', \bar{v}_6')$ is equal to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{a+1} \\ 0 & 0 & 0 & 0 & \frac{1}{a+1} & 0 \\ 0 & 0 & 0 & h_1 & 0 & 0 \\ 0 & 0 & h_1 & 0 & 0 & 0 \\ 0 & \frac{1}{a+1} & 0 & 0 & 0 & 0 \\ \frac{1}{a+1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, there exists a hyperbolic basis $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ of (V, f) such that

$$\bar{v}_1' = \bar{e}_1,\ \bar{v}_6' = \frac{1}{a+1}\bar{f}_1,\ \bar{v}_2' = \bar{e}_2,\ \bar{v}_5' = \frac{1}{a+1}\bar{f}_2,\ \bar{v}_3' = \bar{e}_3,\ \bar{v}_4' = h_1\bar{f}_3.$$

Then:

$$\bar{v}_1 = \bar{e}_1,\ \bar{v}_6 = \frac{1}{a+1}\bar{f}_1,\ \bar{v}_2 = \bar{e}_2,\ \bar{v}_5 = \frac{1}{a+1}\bar{f}_2 + (a+1)h_2\bar{e}_1,\ \bar{v}_3 = \bar{e}_3 + \bar{e}_2,\ \bar{v}_4 = h_1\bar{f}_3 + \bar{e}_1.$$

So,

$$\begin{aligned} \chi &= \frac{a+1}{a} \cdot \bar{v}_1 \wedge \bar{v}_3 \wedge \bar{v}_5 + (a+1) \cdot \bar{v}_2 \wedge \bar{v}_4 \wedge \bar{v}_6 + (\bar{v}_1 + \bar{v}_2) \wedge (\bar{v}_3 + \bar{v}_4) \wedge (\bar{v}_5 + \bar{v}_6) \\ &= \frac{1}{a} \cdot \bar{e}_1 \wedge (\bar{e}_2 + \bar{e}_3) \wedge \bar{f}_2 + \bar{e}_2 \wedge (h_1\bar{f}_3 + \bar{e}_1) \wedge \bar{f}_1 + \frac{1}{a+1} \cdot (\bar{e}_1 + \bar{e}_2) \wedge (\bar{e}_3 + h_1\bar{f}_3) \wedge \\ &\qquad (\bar{f}_1 + \bar{f}_2 + (a+1)^2 h_2 \bar{e}_1). \end{aligned}$$

So, χ is $Sp(V, f)$-equivalent with $\chi_3(h_1, h_2)$.

Reversing the above procedure, we see that the trivector $\chi_3(h_1, h_2)$ is of the form $\phi\Big((\bar{v}_1^* + \delta\bar{v}_2^*) \wedge (\bar{v}_3^* + \delta\bar{v}_4^*) \wedge (\bar{v}_5^* + \delta\bar{v}_6^*)\Big)$ where $\{\bar{v}_1^*, \bar{v}_2^*, \bar{v}_3^*, \bar{v}_4^*, \bar{v}_5^*, \bar{v}_6^*\}$ is some basis of V satisfying $M_{f'}(\bar{v}_1^* + \delta\bar{v}_2^*, \bar{v}_3^* + \delta\bar{v}_4^*, \bar{v}_5^* + \delta\bar{v}_6^*) = M^*$ and

$$M_g(\bar{v}_1^* + \delta\bar{v}_2^*, \bar{v}_3^* + \delta\bar{v}_4^*, \bar{v}_5^* + \delta\bar{v}_6^*) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & h_1(a+1) & 0 \\ 1 & 0 & h_2(a+1) \end{bmatrix}.$$

So, $\chi_3(h_1, h_2)$ is $GL(V)$-equivalent with $\chi_{\mathbb{F}'}^*$ by Lemma 3.4.

We will now determine under which conditions two trivectors of Type (E3') are $Sp(V, f)$-equivalent. Let $h_1, h_2, h_1', h_2' \in \mathbb{F}$ with $h_1 \neq 0 \neq h_1'$. As before, put $k := \frac{1}{a+1}$. The two trivectors $\chi_3(h_1, h_2)$ and $\chi_3(h_1', h_2')$ are $Sp(V, f)$-equivalent if and only if there exists a matrix $A \in SL(3, \mathbb{F}')$ such that

$$M^* = A \cdot M^* \cdot A^{\mathrm{T}}, \tag{1}$$

$$\begin{bmatrix} 0 & 0 & (a+1)k \\ 0 & (a+1)h_1' & 0 \\ (a+1)k & 0 & (a+1)h_2' \end{bmatrix} = A \cdot \begin{bmatrix} 0 & 0 & (a+1)k \\ 0 & (a+1)h_1 & 0 \\ (a+1)k & 0 & (a+1)h_2 \end{bmatrix} \cdot A^{\mathrm{T}} + (a+1) \cdot A \cdot M^* \cdot \mathsf{Im}(A)^{\mathrm{T}}. \tag{2}$$

By (1) and Lemma 3.15, we have that A has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{3}$$

where $a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} \in \mathbb{F}$ such that $a_{22} \cdot a_{33} - a_{23} \cdot a_{32} = 1$. By (2), we have

$$\begin{bmatrix} 0 & 0 & k \\ 0 & h_1' & 0 \\ k & 0 & h_2' \end{bmatrix} = A \cdot \begin{bmatrix} 0 & 0 & k \\ 0 & h_1 & 0 \\ k & 0 & h_2 \end{bmatrix} \cdot A^{\mathrm{T}} + A \cdot M^* \cdot \mathsf{Im}(A)^{\mathrm{T}}. \tag{4}$$

By (3), we have that

$$A \cdot M^* \cdot \mathsf{Im}(A)^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{23} \cdot \mathsf{Im}(a_{22}) + a_{22} \cdot \mathsf{Im}(a_{23}) & a_{23} \cdot \mathsf{Im}(a_{32}) + a_{22} \cdot \mathsf{Im}(a_{33}) \\ 0 & a_{33} \cdot \mathsf{Im}(a_{22}) + a_{32} \cdot \mathsf{Im}(a_{23}) & a_{33} \cdot \mathsf{Im}(a_{32}) + a_{32} \cdot \mathsf{Im}(a_{33}) \end{bmatrix}.$$

By (3), we also have

$$A \cdot \begin{bmatrix} 0 & 0 & k \\ 0 & h_1 & 0 \\ k & 0 & h_2 \end{bmatrix} \cdot A^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & k \\ ka_{23} & a_{22}h_1 & ka_{21} + h_2a_{23} \\ ka_{33} & h_1a_{32} & ka_{31} + h_2a_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{bmatrix}.$$

Comparing the $(1,2)$-entries and the $(1,3)$-entries in both sides of the equality (4), we see that $a_{23} = 0$ and $a_{33} = 1$. Since $a_{22} \cdot a_{33} - a_{23} \cdot a_{32} = 1$, we also have that $a_{22} = 1$. So, we find that

$$A \cdot M^* \cdot \mathsf{Im}(A)^{\mathrm{T}} = \mathrm{diag}(0, 0, \mathsf{Im}(a_{32}))$$

and

$$\begin{aligned} A \cdot \begin{bmatrix} 0 & 0 & k \\ 0 & h_1 & 0 \\ k & 0 & h_2 \end{bmatrix} \cdot A^{\mathrm{T}} &= \begin{bmatrix} 0 & 0 & k \\ 0 & h_1 & ka_{21} \\ k & h_1a_{32} & ka_{31} + h_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & a_{21} & a_{31} \\ 0 & 1 & a_{32} \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & k \\ 0 & h_1 & ka_{21} + h_1a_{32} \\ k & ka_{21} + h_1a_{32} & h_1a_{32}^2 + h_2 \end{bmatrix}. \end{aligned}$$

Equation (4) is then equivalent with

$$\begin{cases} h_1' = h_1, \\ ka_{21} + h_1a_{32} = 0, \\ h_2' = h_2 + h_1a_{32}^2 + \mathsf{Im}(a_{32}). \end{cases}$$

The second equation implies that $a_{21} = \frac{h_1a_{32}}{k}$. The third equation implies that $h_2' = h_2 + h_1(X^2 + aY^2) + Y$, if we put $a_{32} = X + \delta Y$ where $X, Y \in \mathbb{F}$. So, we can conclude:

Proposition 6.2 *Let $h_1, h_2, h_1', h_2' \in \mathbb{F}$ with $h_1 \neq 0 \neq h_1'$. Then the two trivectors $\chi_3(h_1,h_2)$ and $\chi_3(h_1',h_2')$ of V are $Sp(V,f)$-equivalent if and only if $h_1 = h_1'$ and $h_2 + h_2'$ is of the form $h_1(X^2+aY^2)+Y$ for some $X, Y \in \mathbb{F}$.*

Proposition 6.2 is precisely Theorem 1.3(4).

As before, let $h_1, h_2 \in \mathbb{F}$ with $h_1 \neq 0$. By Corollary 3.6, we know that the trivector $\chi_3(h_1,h_2)$ is a trivector of Type (D) when regarded as a trivector of V'. One can now ask to which of the trivectors mentioned in Section 2 $\chi_3(h_1,h_2)$ is $Sp(V',f')$-equivalent to. The following proposition answers this question.

Proposition 6.3 *Let $h_1, h_2 \in \mathbb{F}$ with $h_1 \neq 0$. Then the trivector $\chi_3(h_1,h_2)$ is $Sp(V',f')$-equivalent with γ_1.*

Proof. Put $\chi := \chi_3(h_1,h_2)$. Then

$$\begin{aligned}\chi \;=\; & \Big(\bar{e}_1^*+\delta\bar{e}_2^*\Big)\wedge\Big(\bar{e}_2^*+\bar{e}_3^*+\delta\bar{e}_1^*+\delta h_1\bar{f}_3^*\Big)\wedge\Big(\frac{\bar{f}_2^*}{a(a+1)}+\frac{a+1}{a}h_2\bar{e}_1^*+\frac{\bar{f}_1^*}{a+1}\Big)\\ & +\Big(\bar{e}_2^*+\bar{e}_3^*+\delta\bar{e}_1^*+\delta h_1\bar{f}_3^*\Big)\wedge\Big(\frac{\bar{f}_2^*}{a+1}+(a+1)h_2\bar{e}_1^*+\frac{\delta}{a+1}\bar{f}_1^*\Big)\wedge\Big(\frac{\bar{e}_1^*}{a}+\bar{e}_2^*\Big)\\ & +\Big(\frac{\bar{f}_2^*}{a+1}+(a+1)h_2\bar{e}_1^*+\frac{\delta}{a+1}\bar{f}_1^*\Big)\wedge\Big(\bar{e}_1^*+\delta\bar{e}_2^*\Big)\wedge\Big(\frac{\bar{e}_2^*}{a}+\frac{\bar{e}_3^*}{a}+h_1\bar{f}_3^*+\bar{e}_1^*\Big).\end{aligned}$$

Since $f'(\bar{e}_1^*+\delta\bar{e}_2^*, \bar{e}_2^*+\bar{e}_3^*+\delta\bar{e}_1^*+\delta h_1\bar{f}_3^*) = 0$, $f'(\bar{e}_1^*+\delta\bar{e}_2^*, \frac{\bar{f}_2^*}{a+1}+(a+1)h_2\bar{e}_1^*+\frac{\delta}{a+1}\bar{f}_1^*) = 0$ and $f'(\bar{e}_2^*+\bar{e}_3^*+\delta\bar{e}_1^*+\delta h_1\bar{f}_3^*, \frac{\bar{f}_2^*}{a+1}+(a+1)h_2\bar{e}_1^*+\frac{\delta}{a+1}\bar{f}_1^*) = 1$, the base 3-space of the trivector χ of V' is not totally isotropic. So, χ is $Sp(V',f')$-equivalent with either γ_1 or $\gamma_2(\lambda)$ for some $\lambda \in \mathbb{F}^*$. We also have

$$\begin{aligned}\chi \;=\; & \frac{1}{a}\cdot\bar{e}_1^*\wedge(\bar{e}_2^*+\bar{e}_3^*)\wedge\bar{f}_2^*+\bar{e}_2^*\wedge\bar{f}_1^*\wedge(\bar{e}_1^*+h_1\bar{f}_3^*)\\ & +\frac{1}{a+1}\cdot(\bar{e}_1^*+\bar{e}_2^*)\wedge(\bar{e}_3^*+h_1\bar{f}_3^*)\wedge\Big((a+1)^2h_2\bar{e}_1^*+\bar{f}_1^*+\bar{f}_2^*\Big).\end{aligned}$$

We now compute $\pi(\chi)\wedge\pi(\chi\wedge\pi(\chi))$. We have $\pi(\chi) = \frac{1}{a}\bar{e}_1^*+\bar{e}_2^*$ and

$$\begin{aligned}\chi\wedge\pi(\chi) \;=\; & \frac{h_1}{a}\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_1^*\wedge\bar{f}_3^*+\frac{1}{a}\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge(\bar{e}_3^*+h_1\bar{f}_3^*)\wedge(\bar{f}_1^*+\bar{f}_2^*)\\ & +\frac{1}{a}\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\bar{f}_2^*\wedge\bar{e}_3^*\\ & \frac{1}{a}\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge\Big(\bar{e}_3^*\wedge\bar{f}_1^*+h_1\cdot\bar{f}_3^*\wedge\bar{f}_2^*\Big).\end{aligned}$$

Hence, $\pi(\chi\wedge\pi(\chi)) = \frac{1}{a}\cdot\bar{e}_2^*\wedge\bar{e}_3^*+\frac{h_1}{a}\cdot\bar{e}_1^*\wedge\bar{f}_3^*$ and $\pi(\chi)\wedge\pi(\chi\wedge\pi(\chi)) = \frac{1}{a^2}\cdot\bar{e}_1^*\wedge\bar{e}_2^*\wedge(\bar{e}_3^*+h_1a\bar{f}_3^*)$. Since $\pi(\chi)\wedge\pi(\chi\wedge\pi(\chi))$ is completely decomposable, the trivector $\chi = \chi_3(h_1,h_2)$ must be $Sp(V',f')$-equivalent with γ_1, see Section 4 of [6]. ∎

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